On some basic theoretical problems of fuzzy mathematics

By L. T. Kóczy

Abstract

The aim of this paper is to raise some questions — and partly, also to answer them — in connection with two important problem groups of fuzzy mathematics: n-fuzzy objects and the sigma-properties of different interactive fuzzy structures. These questions are suggested by the analysis of natural languages, the common sense thinking — which are typical fields where the most adequate mathematical model is a fuzzy one —, especially by complex adjectival structures and subjective "verifying" processes, respectively. They have, however, a real practical significance also in the field of engineering, as e.g. in learning machine problems.

In the first part we try to point to the practical importance of the concept of fuzzy objects of type n (or n-fuzzy objects), from the aspect of modelling natural languages. A useful way to define n-fuzzy algebras, i.e., generalizing ordinary fuzzy algebras for n-fuzzy objects, is also given, with introducing an isomorphism mapping from the fuzzy object space to the n-fuzzy object space. As an example, an R-n-fuzzy algebra is defined. Because of the isomorphic property of the above mapping the later studies can be restricted to ordinary fuzzy objects.

In the second part some very basic concepts in connection with the sigma-properties of fuzzy algebras are given and some simple theorems are proved. These are quite important from the aspect of fuzzy learning processes, as their probability theoretic interpretation leads to several convergence theorems — which are not dealt with here, however.

In this part we introduce the concept of the quantified algebra of a fuzzy algebra, and by means of this concept a close relation between interactive fuzzy and Boolean algebras is proved different from the relation between non-interactive system and Boolean algebra given by Zadeh.

Although any presentation of complete application examples is not at all intended in this paper, some aspects of the application of the above results, especially in learning control algorithms, are given, the statements backed up by the experience of a simulation experiment going on at present.

1. Introduction

In the invention of fuzzy mathematics, one of the most important aspects was the intention of obtaining an effective way for modelling badly defined phenomena appearing over and over in the everyday life and many fields of sciences. Modelling by traditional methods proved to be often very coarse. The first problems of this type were raised in pattern recognition by L. A. Zadeh [1, 2]. However, this field turned out soon not to be the one where the application of his new concept made the quickest advance. In disciplines having an even more human factor as linguistics, economics, etc. results could be produced easier. Nevertheless, it did not mean that fuzzy concepts played no part in engineering; rather, that engineering had had its own well-worked-up mathematical background, and to replace it — if only partly — by a new model necessarily met resistance and obstacles. We have to admit it, too, that the bases of fuzzy mathematics have been laid often inexactly which fact resulted, as a matter of course, in the increase of resistance.

As many other, we were suggested by such failures of exactness to look for a correct formulation of fuzziness, by finding and observing real objects and phenomena corresponding to the basic concept of fuzzy objects and operations on them. For this purpose, some phenomena in the common sense thinking and the natural languages turned out to be very suitable. On the basis of these considerations we established a group of fuzzy algebras, among them the most important one was *R*-fuzzy algebra [3, 4].

Some inference methods used in the medical science gave the basic aspects in our constructing these systems; they also pointed to the fact, that the fuzzy operations used formerly (max—min or non-interactive ones) were in accordance only with a restricted part of fuzzy objects. The new type of operations turned out to be much more adequate and applicable as it had been proved by some simple experiments in cluster analysis and learning control carried out by the author and his colleagues.

We now introduce some notions and notations.

I. Pre-R-fuzzy algebra

1. a) There exists a nonempty set

$$X = \{x\},$$

which is named *universe* or *base set.*

b) There exists a nonempty set

$$\otimes = \{A\}$$

the elements of which are named *fuzzy objects.*

c) There exists a mapping M, so that

$$M : \otimes \to \mathscr{F},$$

where $\mathscr{F} = \{\mu | \mu : X \to \mathscr{P}\}$,
where $\mathscr{P}$ is an ordered nonempty set.
(In the practice, the usual representation of $\mathscr{P}$ is the closed interval [0, 1],

then the functions μ and the single values $\mu(x)$ are named membership functions and membership grades, respectively.)

Definition. We define $A=B$ (A equal to B) such that it is the abbreviation for

$$M(A) = M(B).$$

(In the practice it means that

$$\mu_A(x) = \mu_B(x),$$

for all $x \in X$.)

2. a) There exist A and B in $\otimes$ such that $A \neq B$,
   b) $\otimes$ is closed under the binary operation $\vee$, named *disjunction*,
   c) $\otimes$ is closed under the binary operation $\wedge$, named *conjunction*,
   d) $\otimes$ is closed under the unary operation $\neg$, named *negation*.
3. a) $A \vee B = B \vee A$, for all $A, B \in \otimes$.
   b) $(A \vee B) \vee C = A \vee (B \vee C)$, for all $A, B, C \in \otimes$.
   c) $\neg\neg A = A$, for all $A \in \otimes$.
   d) There exists an element $\emptyset \in \otimes$, named zero, such that for all $A \in \otimes$,

$$d1) \quad A \vee \emptyset = A,$$

$$d2) \quad A \wedge \emptyset = \emptyset.$$

   e) $\neg(A \vee B) = \neg A \wedge \neg B$, for all $A, B \in \otimes$.
4. a) $M((A \wedge B) \vee (A \wedge C)) > M(A \wedge (B \vee C))$, for all $A, B, C \in \otimes$, if

$$(A \wedge B) \vee (A \wedge C) \neq \emptyset \quad \text{and} \quad A \wedge (B \vee C) \neq \neg\emptyset.$$

   b) $M((A \vee B) \wedge (A \vee C)) < M(A \vee (B \wedge C))$, for all $A, B, C \in \otimes$, if

$$(A \vee B) \wedge (A \vee C) \neq \neg\emptyset \quad \text{and} \quad A \vee (B \wedge C) \neq \emptyset.$$

   c) $M(A \vee B) > M(A)$, for all $A, B \in \otimes$, if $A \neq \neg\emptyset$ and $B \neq \emptyset$.
   d) $M(A \wedge B) < M(A)$, for all $A, B \in \otimes$, if $A \neq \emptyset$ and $B \neq \neg\emptyset$.
   e) $M(A) - M(B) = M(\neg B) - M(\neg A)$, for all $A, B \in \otimes$.
5. a) If the equation $A \vee U = B$ $(A, B \in \otimes,\ A \neq \neg\emptyset)$ has a solution $U \in \otimes$, than U is unique.
   b) If the equation $A \wedge U = B$ $(A, B \in \otimes,\ A \neq \emptyset)$ has a solution $U \in \otimes$, than U is unique.
6. a) $M(A \vee B)$ is a continuously differentiable function in terms of $M(A)$ and $M(B)$.
   b) $M(A \wedge B)$ is a continuously differentiable function in terms of $M(A)$ and $M(B)$.
   c) $M(\neg A)$ is a continuously differentiable function in terms of $M(A)$.

II. R-fuzzy algebra

1. Like in System I. (Here the usual representation of $\mathscr{P}$ is R^1.)

2—4). Like in System I.

5. a) Equation $A \vee U = B$ $(A, B \in \otimes,\ A \neq \neg\emptyset)$ can be solved for $U \in \otimes$ and U is unique.

b) Equation $A \wedge U = B$ ($A, B \in \otimes$, $A \neq \emptyset$) can be solved for $U \in \otimes$ and U is unique.

6. Like in System I.

III. S-fuzzy algebra

1—3. Like in System I.

4. a) $M(A \vee A) > M(A)$, for all $A \in \otimes$, $A \neq \emptyset$, $A \neq \neg\emptyset$.
   b) $M(A \wedge A) < M(A)$, for all $A \in \otimes$, $A \neq \emptyset$, $A \neq \neg\emptyset$.
   c) $M(A \vee A) > M(B \vee B)$ iff $M(A) > M(B)$; $A, B \in \otimes$.
   d) $M(A \wedge A) > M(B \wedge B)$ iff $M(A) > M(B)$; $A, B \in \otimes$.

5. Like condition 6 in System I.

At the end of the presentation of our axiomatic systems we should like to stress the fact, that all three systems are symmetrical for the operations conjunction and disjunction, that is, they are dual structures — in spite of their asymmetrical appearance.

Although the above systems were constructed merely on the basis of theoretic speculations and passive observations of the rules of common sense inference, after they having been established in a more or less exact form, they would be backed up by some independent, objective, experimental facts discovered by another author [5 and 6, respectively]. Also some independent theoretical considerations pointed to the fact that adequate fuzzy axiomatics must be constructed similarly to I. (Although no dual systems were introduced.) [7]. On the basis of these results we were able to extend the group of representative operation trebles to a very general class (rational functions) [8]. As we are not concerned now with the problem of representatives, we omit a detailed presentation of it.

2. N-fuzzy objects

When fuzzy sets, fuzzy objects were introduced by L. A. Zadeh, they seemed general enough for modelling "all fuzzy-type phenomena of the world" [2]. Later analyses of natural languages etc., however, resulted in the perception of the fact, that a more general concept — modelling twofold, n-fold fuzziness — must be introduced: this has been done also by Zadeh, and has been named "fuzzy object of type n" or more simply "n-fuzzy object" [9]. Although this first formulation was not at all exact, the general idea can be preserved, however, formulating its definition in an entirely different way. As we see it now, this concept is the most general one (having also a practical importance), so we are very much interested in it. In this section we shall present a way, how handling of n-fuzzy objects can be simplified considerably, restricting the field of interest again to ordinary fuzzy systems (as I, II, and III).

According to the axiomatics an ordinary fuzzy object can be given in the following way:

Let X be the given universe, then A_X is as follows:

$$\langle X, qA(x)\rangle,$$

where

$qA: X \to \mathscr{P}$ ($\mathscr{P}$ is [0, 1] in the most simple case.)

Let us consider now an example from the natural language:
"Peter is tall."
and
"Peter is rather tall."
are both fuzzy statements. We are interested now in the predicative part of them. If we define X as

$X = \{$possible values of the height of a man$\}$,

"tall" is a fuzzy set of X (A_X). What is "rather"? It is also a fuzzy expression, but a fuzzy set of *grades*. Let now P be the universe of abstract truth grades mapped on the interval [0, 1]. Then "rather" is a fuzzy set B_P, a double consisting of P and a function $q_B(p)$. We must become aware of the fact, that taking a singular x in X, qA maps it on the ordered set of truth values $\mathscr{P}$, which is identical to P. Thus if we consider an object gained by "modulating" $qA(x)$ by $qB(p)$, we get a mapping $(qA^2(x))$ which maps X on the set of fuzzy subsets of $\mathscr{P}$. This is named 2-fuzzy membership function of the expression "rather tall". A 2-fuzzy object is now the system

$$A_X^2 = \langle X, qA^2(x)\rangle.$$

After this example we can give the exact definition of an n-fuzzy object:

Definition. Given a universe $X = \{x\}$. Then $S_X^n = \langle X, qS^n\rangle$, is an n-fuzzy object of X where

$$qS^n: X \to S_{\mathscr{P}}^{n-1},$$

$$S_{\mathscr{P}}^{n-1} = \langle \mathscr{P}, qS^{n-1}\rangle,$$

$$qS^{n-1}: \mathscr{P} \to S^{n-2},$$

$$\vdots$$

$$S_{\mathscr{P}}^1 = S_{\mathscr{P}} = \langle \mathscr{P}, qS\rangle,$$

$$qS: \mathscr{P} \to \mathscr{P}.$$

A much greater problem is the proper way of defining operations over n-fuzzy objects. It has been done by Zadeh, we must state, however, that his definition is not correct, as the set of n-fuzzy objects is not closed under his disjunction (and conjunction either.) Another way has been taken by Mizumoto *et al.* [10], where no failure of correctness appeared, nevertheless, the properties of n-fuzzy operations contradicted the simplest natural properties of normal operations. We mention only the fact, that in Mizumoto's disjunction, e.g., it is possible that the membership function of the result (taken over one single x) is *less that both* functions of the arguments.

Here, we are going to present a way for defining an arbitrary operation over n-fuzzy objects, which has been introduced over fuzzy objects, in such a way that this definition is always consistent and in accordance with the original one.

Definition. Let $*$ be an arbitrary m-ary operation over fuzzy objects such that

$$*(A_1, A_2, \dots, A_m) = *(\langle X, qA_1\rangle, \langle X, qA_2\rangle, \dots, \langle X, qA_m\rangle) =$$
$$= \langle X, F(qA_1, qA_2, \dots, qA_m)\rangle,$$
$$\text{where} \quad F: M^m \to M, \; M = \{q\},$$
$$\text{and} \quad F(qA_1, qA_2, \dots, qA_m)x = f(qA_1(x), qA_2(x), \dots, qA_m(x)),$$
$$\text{where} \quad f: \mathcal{P}^m \to \mathcal{P}.$$

Let now C_X^n be an n-fuzzy object of X such that

$$C_X^n = \langle X, qC^n\rangle,$$

where

$$qC^n(x) = \langle P_1, qC_x^{n-1}\rangle,$$
$$qC_x^{n-1}(p_1) = \langle P_2, qC_{x,p1}^{n-2}\rangle, \; p_1 \in P_1,$$
$$\vdots$$
$$qC_{x,p1,p2,\dots,p(n-2)}(p_{n-1}) = qC(x, p_1, p_2, \dots, p_{n-1});$$
$$P_1 = P_2 = \dots = P_{n-1} = \mathcal{P}.$$

Let C be the fuzzy "equivalent" of C_X^n:

$$C = \langle Xx\mathcal{P}^{n-1}, qC\rangle = \langle Y, qC\rangle, \quad \text{and} \quad Y = \{y\} = \{(x, p_1, \dots, p_m)\}.$$

Now, $*$ is to be extended in the following way:

Let E be a one-to-one mapping from the set of n-fuzzy objects of X to the set of fuzzy objects of Y such that

$$E(C_X^n) = C.$$

Then

$$*(A_{1X}^n, A_{2X}^n, \dots, A_{mX}^n) = *(\langle X, qA_1^n\rangle, \langle X, qA_2^n\rangle, \dots, \langle X, qA_m^n\rangle) =$$
$$= \langle X, q*(A_1, A_2, \dots, A_m)^n\rangle$$

means, that

$$q*(A_1, A_2, \dots, A_m)^n(y) = f(qE(A_{1X}^n)(y), qE(A_{2X}^n)(y), \dots, qE(A_{mX}^n)(y)).$$

This is equivalent to

$$*(A_{1X}^n, A_{2X}^n, \dots, A_{mX}^n) = E^{-1}(*(E(A_{2X}^n), E(A_{2X}^n), \dots, E(A_{mX}^n))).$$

Although the above definition might seem to be quite complicated at first sight, in reality it is very simple: the operations must be computed point by point in n-fuzzy cases.

On the basis of the above formulae, a very important fact can be proved in a most simple way:

Theorem *(of isomorphism)*. Let $\otimes^n$ be an algebra with r operations of the n-fuzzy objects of X. Moreover, let these r operations be extensions of r similar operations of the algebra $\otimes$ of the fuzzy objects of $X x \mathscr{P}^{n-1} = Y$. If

$$E(S_X^n) = S_Y \quad \text{for all} \quad S_X^n \in \otimes^n,\ S_Y \in \otimes \quad \text{then}$$

$\otimes$ and $\otimes^n$ are isomorphic.

We close this part with a simple example which shows that, using this theorem, the computation in the field of n-fuzzy objects becomes quite simple, proving also that no separate examinations or considerations are necessary when dealing with n-fuzzy problems: All results concerning ordinary fuzzy algebras are also valid for their n-fuzzy equivalents. We shall utilize this fact in the second part.

Example. Let us define disjunction over fuzzy objects as follows:

$$qA \vee B = qA + qB - qA \cdot qB.$$

We represent the interval [0, 1] by 6 points only: 0.0, 0.2, 0.4, 0.6, 0.8, 1.0.

$$X = \{a, b, c, d\}.$$

In the table we give the membership functions of two objects of X: A_X^2 and B_X^2.

	X \ P	0.0	0.2	0.4	0.6	0.8	1.0	
A_X^2:	a	0.0	0.0	0.1	0.3	0.7	0.9	$qA^2(a)$
	b	0.0	0.1	0.2	0.6	1.0	0.6	$qA^2(b)$
	c	0.0	0.2	0.6	1.0	0.9	0.4	$qA^2(c)$
	d	0.2	0.7	1.0	0.9	0.3	0.0	$qA^2(d)$

	X \ P	0.0	0.2	0.4	0.6	0.8	1.0	
B_X^2:	a	0.0	0.3	0.9	1.0	0.7	0.5	$qB^2(a)$
	b	0.2	0.5	0.9	0.9	0.8	0.7	$qB^2(b)$
	c	0.4	0.7	0.9	0.9	0.9	0.9	$qB^2(c)$
	d	0.6	0.9	0.9	0.9	0.9	1.0	$qB^2(d)$

Using the given definition of the disjunction, $C_X^2 = A_X^2 \vee B_X^2$ can be given in the following way

	X \ P	0.0	0.2	0.4	0.6	0.8	1.0	
C_X^2:	a	0.0	0.3	0.91	1.0	0.91	0.95	$qC^2(a)$
	b	0.2	0.55	0.92	0.96	1.0	0.88	$qC^2(b)$
	c	0.4	0.76	0.96	1.0	0.99	0.94	$qC^2(c)$
	d	0.68	0.97	1.0	0.99	0.93	1.0	$qC^2(d)$

3. Basic concepts of fuzzy sigma-algebras

In this section we introduce some very basic ideas in connection with fuzzy sigma-algebras and related problems. Then, we are able to deal with an interesting transformation of algebras named "quantification" which leads to the concept of a special type of convergent algebras.

First we give some definitions.

Definition. G is a fuzzy sigma-algebra, if G is a fuzzy algebra (in either senses of the Introduction, or in the sense, as it has been defined by Zadeh, etc.) and the infinite conjunction and disjunction are defined in G, i.e.,

$$\bigvee_{i\in H} A_i\in G, \quad \text{and} \quad \bigwedge_{i\in H} A_i\in G, \quad \text{where} \quad |H| = \aleph_0, \quad A_i\in G\,\forall\, i.$$

Thus, e.g., III and the sigma-properties together form an S-sigma-algebra.

Definition. Let A_i be a sequence of fuzzy objects

$$A_i = \langle X, qi\rangle \quad (A_i\in G).$$

Then the limit value of A_i is $A=\langle X, q\rangle \quad (A\in G)$,

if
$$q = \lim_{i\to\infty} qi$$

in the ordinary sense. Thus A_i is convergent iff qi is convergent.

Using this notion, the infinite conjunction or disjunction can be computed:

$$\bigvee_{i=1}^{\infty} A_i = \lim_{j\to\infty} B_j, \quad \text{where} \quad B_j = \bigvee_{i=1}^{j} A_i.$$

Definition. $\exists_H$ is named the existential quantifier over the well-ordered set of indices H

$$\exists_H A_i = \bigvee_{i\in H} A_i \quad (A_i\in G).$$

Similarly, the universal quantifier over H, $\forall_H$ is defined as

$$\forall_H A_i = \bigwedge_{i\in H} A_i \quad (A_i\in G).$$

Although this way of generalizing the notion of quantifiers of predicate calculus is not the only (and not at all the most logical) one. However, we shall need this special concept in our further examinations. Since it has a vague resemblance to quantifiers, thus it will not cause any ambiguity.

We mention also, that we will not deal with general case since we are interested only in the special case where

$$A_i = A_j, \quad \text{for all } i, j.$$

Definition. Let Q be the algebra defined in the following way:

The elements of Q are gained from a fuzzy algebra G

$$Q' = \{\exists A\,|\,A\in G\}\cup\{\forall A\,|\,A\in G\},$$

where $\exists A$ and $\forall A$ are used for $\exists_H A_i$ $(A_i = A, \forall i)$ and $\forall_H A_i$ $(A_i = A, \forall i)$, respectively.

Since G is a fuzzy sigma-algebra thus the elements of Q are in G.

The operations in Q' are the same as in G, and are defined in the same way. If Q' is not closed then, it must be completed until it becomes closed. Thus, Q is the minimal closed subalgebra in G containing Q'. In the cases interesting for us we shall see that $Q = Q'$.

Q is named the quantified algebra of G.

4. Quantified algebras

In this section we give some statements concerning the quantified algebras of different fuzzy algebras.

Theorem. Let G be an R-fuzzy sigma-algebra. Then its quantified algebra Q is a Boolean algebra.

Proof. For proving the statement, let us assume, that $X = \{x_0\}$ has only one element. Let K and L be elements of G such that

$$L = \forall K = \bigwedge_{i=1}^{\infty} K.$$

Then

$$L = K \wedge \bigwedge_{i=2}^{\infty} K = K \wedge \forall K = K \wedge L.$$

Thus

$$L = K \wedge L.$$

Thus, because of axiom 4 in II, either $K = T = \neg\emptyset$ or $L = \emptyset$, (And then $K = \emptyset$ too.) In the first case, if $K = T$, then $L = T$. Therefore, taking an arbitrary K, L can be only $\emptyset$ or T.

Similarly, if $M = \exists K$, M is $\emptyset$ or T.

In the case when $X = \{x_0\}$, it can be easily proved now, that Q containes only two elements T and $\emptyset$, since

$$T \vee T = T, T \vee \emptyset = T, \emptyset \vee \emptyset = \emptyset: \quad T \wedge T = T, T \wedge \emptyset = \emptyset, \emptyset \wedge \emptyset = \emptyset;$$

$$\neg T = \emptyset, \ \neg\emptyset = T.$$

Let us consider now the general case, where X is an arbitrary set. Then for an arbitrary $x \in X$,

$$qL(x) = q\emptyset(x_0) \ or \ qT(x) \quad \text{and} \quad qM(x_0) = q\emptyset(x_0) \ or \ qT(x_0).$$

Considering now the ordinary sets and their characteristic functions over X, an isomorphism to the set of all possible functions $qL(x)$ and $qM(x)$, i.e., all membership functions of the elements can be found. The isomorphism mapping orders to one characteristic function the membership function which has the value qT where it is 1, and $q\emptyset$ where it is 0.

Finally, Q has not to be completed, which fact can be proved from the ideas concerning the connection between K and L, and K and M, used in the above part of the proof.

As the characteristic functions of the subsets of a set X form a Boolean algebra, the isomorphic Q is Boolean, too.

Theorem. Let G be an S-algebra, such that cardinality of S is finite. Then the quantified algebra Q of G, is Boolean.

Proof can be found in [8].

For the further examination we introduce some more definitions:

Definition. Let G be a fuzzy sigma-algebra. Then G is named (weakly) sigma-associative, if

$$\bigvee_{i=1}^{\infty} (A_i \vee B_i) = \bigvee_{i=1}^{\infty} A_i \vee \bigvee_{i=1}^{\infty} B_i,$$

and

$$\bigwedge_{i=1}^{\infty} (A_i \wedge B_i) = \bigwedge_{i=1}^{\infty} A_i \wedge \bigwedge_{i=1}^{\infty} B_i, \quad \text{for all} \quad A_i, B_i \in G.$$

Definition. If G is an S-algebra which is sigma-algebra and sigma-associative then we call it an SA-algebra.

Theorem. Let G be an SA-algebra. Then the quantified algebra Q of G is Boolean.

First, we prove two dual lemmata.

Lemma. If $qA > qB$, then $q\exists A \geqq q\exists B$.

Proof. Because of axiom group 4 in III,

$$qA \vee A > qB \vee B.$$

Similarly,

$$q(A \vee A) \vee (A \vee A) > q(B \vee B) \vee (B \vee B),$$

etc. etc.

$$q \bigvee_{i=1}^{\infty} A \geqq q \bigvee_{i=1}^{\infty} B,$$

i.e.

$$q\exists A \geqq q\exists B,$$

which had to be proved.

Lemma. If $qA > qB$, then $q\forall A \geqq q\forall B$.

The proof is the dual of the above one.

Now, we return to the proof of the original statement.

Let A be an arbitrary element different from $\emptyset$ and T, and $B = A \vee A$, $C = \exists A$. (Since G is a sigma-algebra thus $C \in G$.)

Then, obviously,

$$qB \leqq qC, \quad \text{and, since } \exists\exists A = \exists A = C,$$

using the first lemma, we obtain

$$q\exists B = q\bigvee_{i=1}^{\infty} B \leqq qC.$$

Using the property of sigma-associativity, this can be written in the following form

$$q\bigvee_{i=1}^{\infty} A \vee \bigvee_{i=1}^{\infty} A = qC\vee C \leqq qC.$$

Because of axiom group 4 in III, $qC\vee C \geqq qC$, which means, that

$$C = C\vee C.$$

Restricting our considerations to sets X with one single element, similarly as in the former proof, it can be seen, that C can be only T or $\emptyset$. That means, that if allowing arbitrary sets X, the elements of Q are in isomorphism with the ordinary subsets of X (the dual proof for $D=\forall A$ must, as a matter of course, added to the above one), thus Q is a Boolean algebra.

Without going into details, we mention, that some related definitions (strong sigma-associativity and sigma-continuity) are dealt with in [8]; on the basis of them different theorems can be proved, which give some practical conditions for a fuzzy algebra to be sigma-associative, etc.

Here we give only one statement yet, referring to Zadeh's so-called non-interactive algebra.

Theorem. If G is a non-interactive fuzzy sigma-algebra, (i.e. a distributive lattice), the quantified algebra Q of G is identical to G. (The very simple proof can also be found in [8].)

Finally, we should stress again the fact, that — because of the examinations in Section 2 — all the above theorems hold for arbitrary n-fuzzy structures, as well.

5. On the application possibilities

Here, we do not intend to deal with application possibilities in detail, only some very general aspects will be given.

Let us assume, that we have a concrete representant of an — e.g. R-fuzzy — algebra, as an example, we present here the most simple and well-known one: the so-named "soft definitions", as follows

$$qA\vee B = qA + qB - qAqB,$$

$$qA\wedge B = qAqB,$$

$$q\neg A = 1 - qA.$$

(A much more general discussion of representants can be found in [8].)

We intend to use this fuzzy calculus for realizing learning algorithms. A very obvious way for this is the formulation of a learning fuzzy automaton which acts on behalf of a membership function (or more than one functions) representing a present strategy for the automaton. The membership function itself is gained by a learning generative process, where fuzzy calculus has a basic role. We have a system that we want to control by our automaton. In a particular situation (element of X, now the situation space) the automaton produces a particular output. (Let us assume now, that the automaton is the fuzzy generalization of a bang-bang type controller, i.e., it has two possible outputs, 1 and 0, and — being a fuzzy one — something like "meanthings" between this two, the values of the interval [0, 1].)

It can be proved in the case of some special optimality criteria, that the best output is the maximal possible value (or the minimal) in a particular situation. Then a given output of the automaton can be more or less near to this or, in contrary, more or less far from it. In the first case a little "quantum" of membership function will be "added" (i.e. disjoined) to the given function such that it converges to the maximal value, and in the second case, it is added in such a way that it converges to the other extremum. Then, if the learning process is going on, it can be hoped, that the values of the membership function(s) over all x-es will be near to the optimal extremum.

Now, we do not intend to deal with the problem, how a fuzzy automaton executes a fuzzy instruction. Let us assume, that both possible outputs have one membership function, that we hope to converge to 1 over such x's where an output 1 or 0 is needed, respectively; and to 0, where 0 or 1, respectively. (Thus two functions belong to the outputs "1" and "0", respectively.)

The real mathematical problem is here: Whether this algorithm converges in the exact mathematical sense?

Theorem. If C is a calculus representing G, an R-fuzzy algebra, the disjunction in C is $\vee$, and the inverse operation of it (which can be defined on the basis of axiom group 5 in II) is denoted by $\vee^-$, f is an indicator taking the value 1, if a strategy S is "good" (in any sense), and 0, if it is "bad", then the following algorithm generating the membership function $qS(x)$ of the strategy S is convergent over all x, where the expected value of f is greater than 0

$$qS(x)_0 = 0,$$

$$qS'(x)_{n+1} = \quad \mathit{if}\ f = 1 \quad \mathit{then} \quad qS'(x) \vee q(x) \quad \mathit{else} \quad qS'(x) \vee^- q(x),$$

$$qS(x)_{n+1} = \quad \mathit{if}\ qS'(x)_n \geqq 0 \quad \mathit{then} \quad qS'(x)_n \quad \mathit{else} \quad 0.$$

$q(x)$ is chosen such that $q(x)$ is greater than a positive ε and less than 1 over x, and 0 elsewhere.

Here, we do not prove this statement.

Using other types of $q(x)$, similar theorems can be proved (however, in the case of not so general conditions). The same is the case, when G is not R-fuzzy, but SA-algebra. Then, however, the algorithm must slightly be changed.

Finally, we mention, that we have always some experimental results using an algorithm similar to the above one (controlled is a stochastic, double integrating system).

DEPT. OF COMMUNICATION ELECTRONICS,
TECHNICAL UNIVERSITY OF BUDAPEST
H-1111 BUDAPEST, HUNGARY
STOCZEK U. 2. ST ÉP.

References

[1] BELLMAN, R. E., R. KALABA, L. A. ZADEH, Abstraction and pattern classification, *J. Math. Anal. Appl.*, v. 13, 1966, pp. 1—7.
[2] ZADEH, L. A., Fuzzy sets, *Information and Control*, v. 8, 1965, pp. 338—353.
[3] KÓCZY, L. T., A fuzzy halmazok néhány elméleti kérdése (Some theoretical questions of fuzzy of sets), *Report, Dept. of Proc. Contr.*, Tech. Univ. Budapest, Jan. 1974.
[4] KÓCZY, L. T., M. HAJNAL, A new fuzzy calculus and its application as a pattern recognition technique, Modern Trends of Cybernetics and General Systems, *Proc. of the 3rd Congress of WOGSC*, Bucharest, Aug. 1975, pp. 66—81.
[5] KÓCZY, L. T., A fuzzy halmazok elmélete és műszaki alkalmazásai (The theory of fuzzy sets and their applications in technology), *Report, Dept. of Proc. Contr.*, Tech. Univ. Budapest, May 1975.
[6] HAMACHER, H., Über logische Verknüpfungen unscharfer Aussagen und deren zugehörige, Bewertungsfunktionen, Arbeitsbericht, *Lehrst. für Unternehmensforschung RWTH* Aachen, July 1975.
[7] RÖDDER, W., On "and" and "or" connective in fuzzy set theory, *WP Lst. für Unt. forsch. RWTH* Aachen, Jan. 1975.
[8] KÓCZY, L. T., Fuzzy algebrák és műszaki alkalmazásaik néhány kérdése (Fuzzy algebras and some questions of their technical applications), *PhD thesis, Dept. of Proc. Contr.*, Tech. Univ. Budapest, May 1976.
[9] ZADEH, L. A., The concept of liguistic variable and its application to approximate reasoning I, *Information Sci.*, v. 8, 1975, pp. 199—249.
[10] SELYE, H., *From dream to discovery on being a scientist*, McGraw Hill Book Co.; 2nd Hung. ed. Akadémiai Kiadó, Budapest, 1974.
[11] MIZUMOTO, M. & K. TANAKA, Some properties of fuzzy sets of type 2, *Information and Control.*, v. 31, 1976, pp. 312—340.

(Received Feb. 16, 1977)