On α_i-products of automata

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The purpose of this paper is to study the α_i-products (see [1]) from the point of view of isomorphic completeness. Namely, we give necessary and sufficient conditions for a system of automata to be isomorphically complete with respect to the α_i-product. It will turn out that there exists no minimal isomorphically complete system of automata with respect to α_i-product and if $i \geqq 1$ then isomorphically complete systems coincide with each other with respect to different α_i-products. Moreover, we prove that if $i<j$ then the α_j-product is isomorphically more general than the α_i-product.

By an automaton we mean a finite automaton without output. Let $\mathbf{A}_t = (x_t, A_t, \delta_t)$ $(t=1, \dots, n)$ be a system of automata. Moreover, let X be a finite nonvoid set and φ a mapping of $A_1 \times \dots \times A_n \times X$ into $X_1 \times \dots \times X_n$ such that $\varphi(a_1, \dots, a_n, x) = (\varphi_1(a_1, \dots, a_n, x), \dots, \varphi_n(a_1, \dots, a_n, x))$, and each φ_j $(1 \leqq j \leqq n)$ is independent of states having indices greater than or equal to $j+i$, where i is a fixed nonnegative integer. We say that the automaton $\mathbf{A} = (X, A, \delta)$ with $A = A_1 \times \dots \times A_n$ and

$$\delta((a_1, \dots, a_n), x) = (\delta_1(a_1, \varphi_1(a_1, \dots, a_n, x)), \dots, \delta_n(a_n, \varphi_n(a_1, \dots, a_n, x)))$$

is the *α_i-product* of $\mathbf{A}_t$ $(t=1, \dots, n)$ with respect to X and φ. For this product we use the shorter notation $\mathbf{A} = \prod_{t=1}^{n} \mathbf{A}_t(X, \varphi)$.

Let Σ be a system of automata. Σ is called *isomorphically complete* with respect to the α_i-product if any automaton can be embedded isomorphically into an α_i-product of automata from Σ. Furthermore, Σ is called *minimal* isomorphically complete system if Σ is isomorphically complete and for arbitrary $\mathbf{A} \in \Sigma$ the system $\Sigma \setminus \{\mathbf{A}\}$ is not isomorphically complete.

Take a set M of automata, and let i be an arbitrary nonnegative integer. Let $\alpha_i(M)$ denote the class of all automata which can be embedded isomorphically into an α_i-product of automata from M. It is said that the α_i-product is *isomorphically more general* than the α_j-product if for any set M of automata $\alpha_j(M) \subseteq \alpha_i(M)$ and there exists at least one set $\bar{M}$ such that $\alpha_j(\bar{M})$ is a proper subclass of $\alpha_i(\bar{M})$.

The following statement is obvious for arbitrary natural number $i \geqq 0$.

Lemma. If **A** can be embedded isomorphically into an α_i-product **B** with a single factor and **B** can be embedded isomorphically into an α_i-product **C** with a single factor, then **A** can be embedded isomorphically into an α_i-product **C** with a single factor.

For any natural number $n \geqq 1$ denote by $\mathbf{T}_n = (T_n, N, \delta_N)$ the automaton for which $N = \{1, \dots, n\}$, T_n is the set of all transformations t of N, and $\delta_N(j, t) = t(j)$ for all $j \in N$ and $t \in T_n$.

The next Theorem gives necessary and sufficient conditions for a system of automata to be isomorphically complete with respect to α_0-product.

Theorem 1. A system Σ of automata is isomorphically complete with respect to α_0-product if and only if for any natural number $n \geqq 1$, there exists an automaton $\mathbf{A} \in \Sigma$ such that $\mathbf{T}_n$ can be embedded isomorphically into an α_0-product of **A** with a single factor.

Proof. The necessity and sufficiency of these conditions will be proved in a similar way as that of the corresponding statement for generalized α_0-product in [2].

In order to prove the necessity assume that Σ is isomorphically complete with respect to the α_0-product. Let $n > 1$ be a natural number and take $\mathbf{T}_n$. By our assumption, $\mathbf{T}_n$ can be embedded isomorphically into an α_0-product $\mathbf{B} = (T_n, B, \delta_\mathbf{B}) = \prod_{t=1}^{m} \mathbf{A}_t(T_n, \varphi)$ of automata from Σ. Assume that $m > 1$, and let μ denote a suitable isomorphism. Define partitions π'_j $(j = 1, \dots, m)$ on B in the following way: $(a_1, \dots, a_m) \equiv (a'_1, \dots, a'_m)(\pi'_j)$ $(a_1, \dots, a_m)$, $(a'_1, \dots, a'_m) \in B$ if and only if $a_1 = a'_1, \dots, a_j = a'_j$. Now let π_j $(j = 1, \dots, m)$ be partitions on N given as follows: for any $(a_1, \dots, a_m), (a'_1, \dots, a'_m) \in B$ we have $\mu^{-1}(a_1, \dots, a_m) \equiv \mu^{-1}(a'_1, \dots, a'_m)(\pi_j)$ if and only if $(a_1, \dots, a_m) \equiv (a'_1, \dots, a'_m)$ (π'_j). It is easy to prove that π_j $(j = 1, \dots, m)$ have the Substitution Property (SP). On the other hand, for $\mathbf{T}_n$ only the two trivial partitions have SP. Thus, we get that each π_j has one-element blocks only, or it has one block only. Among these partitions there should be at least one which has more than one block, since $n > 1$. Let l be the least index for which π_l has at least two bloks. Then the blocks of π_l consist of single elements. Therefore, the number of all blocks of π_l is n. We show that $\mathbf{T}_n$ can be embedded isomorphically into an α_0-product $\mathbf{A}_l$ with a single factor. Let $(a_{i1}, \dots, a_{im})$ denote the image of i $(i = 1, \dots, n)$ under μ. From our assumption and the definition of π_j it follows that $a_{ks} = a_{1s}$ if $1 \leqq k \leqq n$ and $1 \leqq s \leqq l-1$. Take the α_0-product $\mathbf{C} = (T_n, A_l, \delta_\mathbf{C}) = \Pi \mathbf{A}_l(T_n, \Psi)$ where $\Psi(t) = \varphi_l(a_{11}, \dots, a_{1l-1}, t)$ for all $t \in T_n$. It is easy to prove that mapping $\nu: i \to a_{il}$ $(i = 1, \dots, n)$ is an isomorphism of $\mathbf{T}_n$ into $\mathbf{C} = \Pi \mathbf{A}_l(T_n, \Psi)$.

The case $n = 1$ is obvious.

To prove the sufficiency take an automaton $\mathbf{A} = (X, A, \delta_\mathbf{A})$ with n states. Let μ be an arbitrary $1-1$ mapping of A onto N. Take the α_0-product $\mathbf{C} = \Pi \mathbf{T}_n(X, \varphi)$ with a single factor, where $\varphi(x) = t$ if and only if $\mu(\delta_\mathbf{A}(a, x)) = t(\mu(a))$ for any $a \in A$. Then μ is an isomorphism of **A** into **C**. On the other hand, by our assumption, there exists an automaton **B** in Σ such, that $\mathbf{T}_n$ can be embedded isomorphically into an α_0-product of **B** with a single factor. Therefore, by our Lemma, **A** can be embedded isomorphically into an α_0-product of **B**, which completes the proof of Theorem 1.

Corollary. There exists no system of automata which is isomorphically complete with respect to α_0-product and minimal.

Proof. Take a system Σ of automata which is isomorphically complete with respect to α_0-product, and let $\mathbf{A}\in\Sigma$ be an automaton with n states. It is obvious that $\mathbf{A}$ can be embedded isomorphically into an α_0-product of $\mathbf{T}_m$ with a single factor if $m\geqq n$. Take a natural number $m>n$. By Theorem 1, there exists a $\mathbf{B}\in\Sigma$ such that $\mathbf{T}_m$ can be embedded isomorphically into an α_0-product of $\mathbf{B}$ with a single factor. Therefore, by our Lemma, $\mathbf{A}$ can be embedded isomorphically into an α_0-product of $\mathbf{B}$ with a single factor. Thus, $\Sigma\setminus\{\mathbf{A}\}$ is isomorphically complete with respect to α_0-product, showing that Σ is not minimal.

For any natural number $n\geqq 1$ denote by $\mathbf{D}_n=(\{x_{pq}\}_{\substack{1\leqq p\leqq n\\1\leqq q\leqq n}}, \{1, \ldots, n\}, \delta_n)$ the automaton for which for any $l\in\{1, \ldots, n\}$ and $x_{sk}\in\{x_{pq}\}$

$$\delta_n(l, x_{sk}) = \begin{cases} k & \text{if } l = s \\ l & \text{otherwise.}\end{cases}$$

The following Theorem holds for α_i-products with $i\geqq 1$.

Theorem 2. A system Σ of automata is isomorphically complete with respect to α_i-product ($i\geqq 1$) if and only if for any natural number $n\geqq 1$, there exists an automaton $\mathbf{A}\in\Sigma$ such that $\mathbf{D}_n$ can be embedded isomorphically into an α_i-product of $\mathbf{A}$ with a single factor.

Proof. First we prove that $\mathbf{D}_n$ $(n>1)$ can be embedded isomorphically into an α_i-product of automata from Σ with at most i factors if $\mathbf{D}_n$ can be embedded isomorphically into an α_i-product of automata from Σ. Indeed, assume that $\mathbf{D}_n$ can be embedded isomorphically into the α_i-product $\mathbf{B}=\prod_{t=1}^{k}\mathbf{A}_t(\{x_{pq}\}, \varphi)$ of automata from Σ with $k>i$, and let μ denote the isomorphism. For any $l\in\{1, \ldots, n\}$ denote by $(a_{l1}, \ldots, a_{lk})$ the image of l under μ. We may suppose that there exist natural numbers $r\neq s$ $(1\leqq r, s\leqq n)$ such that $a_{r1}\neq a_{s1}$ since otherwise $\mathbf{D}_n$ can be embedded isomorphically into an α_i-product of automata from Σ with $k-1$ factors. Now assume that there exist natural numbers $u\neq v$ $(1\leqq u, v\leqq n)$ such that $a_{ut}=a_{vt}$ $(t=1, \ldots, i)$. Then $\varphi_1(a_{u1}, \ldots, a_{ui}, x_{lr})=\varphi_1(a_{v1}, \ldots, a_{vi}, x_{lr})$ for any $x_{lr}\in\{x_{pq}\}$. Thus in the α_i-product $\mathbf{B}$ the automaton $\mathbf{A}_1$ obtains the same input signal in the states a_{u1} and a_{v1} for any $x_{lr}\in\{x_{pq}\}$. On the other hand since μ is an isomorphism and $u\neq v$, thus the automaton $\mathbf{A}_1$ from the state a_{u1} goes into the state a_{r1} and from the state a_{v1} it goes into the state a_{v1} for any input signal x_{ur} $(1\leqq r\leqq n)$. This implies $a_{v1}=a_{r1}$ $(1\leqq r\leqq n)$, which contradicts our assumption. Thus we get that the elements $(a_{t1}, \ldots, a_{ti})$ $(1\leqq t\leqq n)$ are pairwise different. Take the following α_i-product $\mathbf{C}=(\{x_{pq}\}, C, \delta_c)=\prod_{t=1}^{i}\mathbf{A}_t(\{x_{pq}\}, \psi)$ where for any $j=1, \ldots, i$, $(a_1, \ldots, a_i)\in A_1\times\ldots\times A_i$ and $x\in\{x_{pq}\}$

$$\psi_j(a_1, \ldots, a_i, x) = \begin{cases} \varphi_j(a_{t1}, \ldots, a_{tj+i-1}, x) & \text{if } j+i-1\leqq k \text{ and there exists} \\ & 1\leqq t\leqq n \text{ such that } a_s=a_{ts}\ (s=1, \ldots, i,) \\ \varphi_j(a_{t1}, \ldots, a_{tk}, x) & \text{if } j+i-1>k \text{ and there exists} \\ & 1\leqq t\leqq n \text{ such that } a_s=a_{ts}\ (s=1, \ldots, i), \\ \text{arbitrary input signal from } X_j & \text{otherwise.}\end{cases}$$

It is clear that the correspondence $\nu: l \to (a_{l1}, \ldots, a_{li})$ is an isomorphism of $\mathbf{D}_n$ into $\mathbf{C}$.

Now we show that if $\mathbf{D}_n$ $(n>1)$ can be embedded isomorphically into an α_i-product of automata from Σ with at most i factors then there exists an automaton $\mathbf{A} \in \Sigma$ such that $\mathbf{D}_{[\sqrt[i]{n}]}$ can be embedded isomorphically into an α_i-product of $\mathbf{A}$ with a single factor, where $[\sqrt[i]{n}]$ denotes the largest integer less than or equal to $\sqrt[i]{n}$. Indeed, assume that $\mathbf{D}_n$ can be embedded isomorphically into the α_i-product $\mathbf{B} = \prod_{t=1}^{k} \mathbf{A}_t(\{x_{pq}\}, \varphi)$ of automata from Σ with $k \leqq i$ factors. Let μ denote a suitable isomorphism, and for any $l \in \{1, \ldots, n\}$ let $(a_{l1}, \ldots, a_{lk})$ be the image of l under μ. Since μ is a $1-1$ mapping, thus the elements $(a_{t1}, \ldots, a_{tk})$ $(t=1, \ldots, n)$ are pairwise different. Therefore, there exists an s $(1 \leqq s \leqq k)$ such that the number of pairwise different elements among $a_{1s}, a_{2s}, \ldots, a_{ns}$ is greater than or equal to $[\sqrt[i]{n}]$. Let $a_{j_1 s}, \ldots, a_{j_r s}$ denote pairwise different elements, where $r \geqq [\sqrt[i]{n}]$, and denote by $\bar{X}$ the set of input signals x_{pq} $(1 \leqq p, q \leqq [\sqrt[i]{n}])$. Take the following α_i-product $\mathbf{C} = \prod \mathbf{A}_s(\bar{X}, \Psi)$ with single factor, where for any $a_{j_t s} \in \mathbf{A}_s$ and $x_{uv} \in \bar{X}$

$$\psi(a_{j_t s}, x_{uv}) = \begin{cases} \varphi_s(a_{j_t 1}, \ldots, a_{j_t k}, x_{j_t j_v}) & \text{if} \quad u = t \\ \varphi_s(a_{j_t 1}, \ldots, a_{j_t k}, x_{j_t j_t}) & \text{otherwise.} \end{cases}$$

It can be proved easily that the correspondence $\nu: t \to a_{j_t s}$ $(t=1, \ldots, [\sqrt[i]{n}])$ is an isomorphism of $\mathbf{D}_{[\sqrt[i]{n}]}$ into $\mathbf{C}$.

The case $n=1$ is again obvious. To prove the sufficiency by our Lemma, it is enough to show that arbitrary automaton with n states can be embedded isomorphically into an α_i-product of $\mathbf{D}_n$ with a single factor. This is trivial.

Corollary. There exists no system of automata which is isomorphically complete with respect to α_i-product $(i \geqq 1)$ and minimal.

In the sequel we shall study general properties of α_i-products $(i=0, 1, \ldots)$. For this we need some preparation.

Take a set A and a system $\pi_0, \ldots, \pi_n$ of partitions on A. We say that this system of partitions is *regular* if the following conditions are satisfied:

(1) π_0 has one block only,

(2) π_n has one-element blocks only,

(3) $\pi_0 \geqq \pi_1 \geqq \ldots \geqq \pi_n$.

Let π be a partition of A. For any $a \in A$, denote by $\pi(a)$ the block of π containing a. Moreover, set $M_{j,a} = \{\pi_{j+1}(b) : b \in A \text{ and } b \equiv a(\pi_j)\}$, where $a \in A$ and $j=0, \ldots, n-1$. Finally, let $\pi_j/\pi_{j+1} = \max \{|M_{j,a}| : a \in A\}$.

It holds the following.

Theorem 3. Let $l>2$ be a natural number and $i \geqq 1$. An automaton $\mathbf{A} = (X, A, \delta_{\mathbf{A}})$ can be embedded isomorphically into an α_i-product of automata having fewer states than l, if and only if there exists a regular system $\pi_0, \ldots, \pi_n$ of partitions of A such that

(I) $\pi_j/\pi_{j+1} < l$ for all $j=0, \ldots, n-1$,

(II) $a \equiv b(\pi_j)$ implies $\delta_A(a, x) = \delta_A(b, x)$ (π_{j-i+1}) for all $i-1 \leqq j \leqq n$, $x \in X$ and $a, b \in A$.

Proof. Theorem 3 will be proved in a similar way as the corresponding statement for generalized α_i-products in [2].

In order to prove necessity assume that the automaton **A** can be embedded isomorphically into an α_i-product $\prod_{t=1}^{n} \mathbf{A}_t(X, \varphi)$ of automata with $|A_t| < l$ $(t=1, \ldots, n)$ and $l>2$. Let μ denote a suitable isomorphism. Define partitions π_j $(j=0, 1, \ldots, n)$ on A in the following way: π_0 has one block only, and $a \equiv a'(\pi_j)$ $(1 \leqq j \leqq n)$ if and only if $\mu(a) = (a_1, \ldots, a_n)$, $\mu(a') = (a'_1, \ldots, a'_n)$ and $a_1 = a'_1, \ldots, a_j = a'_j$. It is obvious that $\pi_0, \pi_1, \ldots, \pi_n$ is a regular system of partitions and conditions (I) and (II) are satisfied by this system.

Conversely, assume that for an $\mathbf{A} = (X, A, \delta)$ there exists a regular system $\pi_0, \ldots, \pi_n$ of partitions satisfying conditions (I) and (II). We construct automata $\mathbf{A}_j = (X_j, A_j, \delta_j)$ $(j=1, \ldots, n)$ with $|A_j| = \pi_{j-1}/\pi_j (<l)$ such that the automaton **A** can be embedded isomorphically into an α_i-product of automata $\mathbf{A}_j$ $(j=1, \ldots, n)$.

Let A_j be arbitrary abstract sets with $|A_j| = \pi_{j-1}/\pi_j$ and $X_j = A_1 \times \ldots \times A_{j+i-1} \times X$ if $j+i-1 \leqq n$ and $X_j = A_1 \times \ldots \times A_n \times X$ otherwise. Now let μ_j be a mapping of $M_j = \{\pi_j(a) : a \in A\}$ onto A_j such that the restriction of μ_j to any $M_{j-1,a}$ is $1-1$. Define the transition function δ_j in the following way:

(1) if $j+i-1 \leqq n$ then for any $a_j \in A_j$ and $(b_1, \ldots, b_{j+i-1}, x) \in X_j$

$$\delta_j(a_j, (b_1, \ldots, b_{j+i-1}, x)) = \begin{cases} \mu_j(\pi_j(\delta(a, x))) \text{ if } a_j = b_j \text{ and there exists an } a \in A \\ \qquad \text{such that } \mu_t(\pi_t(a)) = b_t \text{ for all } t=1, \ldots, i+j-1, \\ \text{arbitrary element from } A_j \text{ otherwise,} \end{cases}$$

(2) if $j+i-1 > n$ then for any $a_j \in A_j$ and $(b_1, \ldots, b_n, x) \in X_j$

$$\delta_j(a_j, (b_1, \ldots, b_n, x)) = \begin{cases} \mu_j(\pi_j(\delta(a, x))) \text{ if } a_j = b_j \text{ and there exists an } a \in A \\ \qquad \text{such that } \mu_t(\pi_t(a)) = b_t \text{ for all } t=1, \ldots, n, \\ \text{arbitrary element from } A_j \text{ otherwise.} \end{cases}$$

First we prove that δ_j is well defined. Assume that in case (1) there exists a $b \in A$ such that $\mu_t(\pi_t(b)) = b_t$ $(t=1, \ldots, j+i-1)$. It is enough to show that $b \equiv a(\pi_{j+i-1})$ since this by (II), implies that $\delta(b, x) \equiv \delta(a, x)$ for any $x \in X$. We proceed by induction on t. $b \equiv a(\pi_1)$ obviously holds since μ_1 is a $1-1$ mapping of M_1 onto A_1. Assume that our statement has been proved for $t-1$ $(1 \leqq t-1 < j+i-1)$ that is $b \equiv a(\pi_{t-1})$. Therefore, since μ_t is $1-1$ on $M_{t-1,a}$ and $\mu_t(\pi_t(a)) = \mu_t(\pi_t(b))$ thus $\pi_t(b) = \pi_t(a)$. Case (2) can be proved by a similar argument.

Take the α_i-product $\mathbf{B} = \prod_{t=1}^{n} \mathbf{A}_t(X, \varphi)$ where the mapping φ_j is defined in the following way:

(1) if $j+i-1 \leqq n$ then for any $(a_1, \ldots, a_{j+i-1}) \in A_1 \times \ldots \times A_{j+i-1}$ and $x \in X$

$$\varphi_j(a_1, \ldots, a_{j+i-1}, x) = (a_1, \ldots, a_{j+i-1}, x),$$

(2) if $j+i-1 > n$ then for any $(a_1, \ldots, a_n) \in A_1 \times \ldots \times A_n$ and $x \in X$

$$\varphi_j(a_1, \ldots, a_n, x) = (a_1, \ldots, a_n, x).$$

It is easy to prove that the mapping $\nu: a \to (\mu_1(\pi_1(a)), \ldots, \mu_n(\pi_n(a)))$ is an isomorphism of $\mathbf{A}$ into $\mathbf{B}$, which completes the proof of Theorem 3.

Let us denote by $\mathbf{A}_2 = (\{x, y\}, \{0, 1\}, \delta_2)$ the automaton for which $\delta_2(0, x) = \delta_2(1, y) = 1$ and $\delta_2(1, x) = \delta_2(0, y) = 0$.

Now we prove

Theorem 4. Automaton $\mathbf{D}_n$ can be embedded isomorphically into an α_i-product of $\mathbf{A}_2$ $(i \geqq 1)$ if and only if $1 \leqq n \leqq 2^i$.

Proof. The necessity follows from Theorem 3. Indeed, if $\mathbf{D}_n$ can be embedded isomorphically into an α_i-product of $\mathbf{A}_2$, then by Theorem 3, there exists a regular system $\pi_0, \pi_1, \ldots, \pi_k$ of partitions of the set $\{1, \ldots, n\}$ such that (I) and (II) are satisfied. If $n > 2^i$ then there exists a subsystem $\pi_{t_1} > \pi_{t_2} > \ldots > \pi_{t_i}$ of $\pi_0, \ldots, \pi_k$ such that $\pi_0 > \pi_{t_1}$ and $\pi_{t_i} > \pi_k$. Since $\pi_{t_i} > \pi_k$ thus there exists at least one block of π_{t_i} which has more than one element, that is there exist l and r $(1 \leqq l, r \leqq n)$ with $l \neq r$ and $l \equiv r(\pi_{t_i})$. From this, by condition (II), we get that for all $x_{sv} \in \{x_{pq}\}_{\substack{1 \leqq p \leqq n \\ 1 \leqq q \leqq n}}$ $\delta_n(l, x_{sv}) \equiv \delta_n(r, x_{sv})(\pi_{t_1})$. This implies $\pi_0 = \pi_{t_1}$, which contradicts the assumption that $\pi_0 > \pi_{t_1}$.

To prove the sufficiency let n be an arbitrary natural number with $1 \leqq n \leqq 2^i$. We take the α_i-product $\mathbf{B} = \prod_{t=1}^{i} \mathbf{A}_2(\{x_{pq}\}, \varphi)$ of $\mathbf{A}_2$, where the mapping φ_j is defined in the following way: for any

$$(a_1, \ldots, a_i, x_{sr}) \in \{0, 1\} \times \{0, 1\} \times \ldots \times \{0, 1\} \times \{x_{pq}\}$$

$$\varphi_j(a_1, \ldots, a_i, x_{sr}) = \begin{cases} x & \text{if } \sum_{t=1}^{i} a_t 2^{i-t} + 1 = s \text{ and } r = \sum_{t=1}^{i} b_t 2^{i-t} + 1 \text{ and } a_j \neq b_j, \\ y & \text{otherwise.} \end{cases}$$

It is not difficult to prove that $\mathbf{D}_n$ can be embedded isomorphically into the automaton $\mathbf{B}$ under the isomorphism μ defined as follows: if $k = \sum_{t=1}^{i} a_t 2^{i-t} + 1$ then $\mu(k) = (a_1, \ldots, a_i)$ for all $k = 1, \ldots, n$. This ends the proof of Theorem 4.

Let $\mathbf{C}_n$ denote the automaton $(\{x\}, \{1, \ldots, n\}, \delta_n)$ where for all $1 \leqq k < n$ $\delta_n(k, x) = k+1$ and $\delta_n(n, x) = n$.

It can easily be seen that for any natural number $n \geqq 1$ $\mathbf{C}_n$ can be embedded isomorphically into an α_1-product of $\mathbf{A}_2$. On the other hand it is not difficult to prove that if $n > 1$ then $\mathbf{C}_n$ cannot be embedded isomorphically into an α_0-product of $\mathbf{A}_2$. From this we obtain that the α_1-product is isomorphically more general than the α_0-product.

In [3] V. M. Gluskov introduced the concept of the general product and proved that system $\{\mathbf{A}_2\}$ is isomorphically complete with respect to the general product. This, by Theorem 4, implies that for any natural number i the general product is isomorphically more general than the α_i-product.

Our results can be summarized by

Theorem 5. The general product is isomorphically more general than any α_j-product $(j = 0, 1, 2, \ldots)$ and any i, j $(i, j \in \{0, 1, 2, \ldots\})$ if $i < j$ then the α_j-product is isomorphically more general than the α_i-product.

Finally we consider that what kind automata can be embedded isomorphically into an α_i-product $(i=0, 1, 2, \ldots)$ of automata from the given finite set of automata. For this the following is valid.

Theorem 6. For any natural number $i(\geqq 0)$, automaton $\mathbf{A}$ and finite set M of automata it can be decided whether or not $\mathbf{A}\in\alpha_i(M)$.

Proof. Assume that automaton $\mathbf{A}=(X, A, \delta_{\mathbf{A}})$ with m states can be embedded isomorphically into an α_i-product $\mathbf{B}=\prod_{t=1}^{s}\mathbf{A}_t(X, \varphi)$ of automata from M under the isomorphism μ. Let $V=\max\{|A_t|:\mathbf{A}_t\in M\}$, and for all $a_i\in A$ $(i=1, \ldots, m)$ denote by $(a_{i1}, \ldots, a_{is})$ the image of a_i under μ. We define partition π on the set of indices of the α_i-product $\mathbf{B}$. Any k, l $(1\leqq k, l\leqq s)$ $k\equiv l(\pi)$ if and only if $\mathbf{A}_k=\mathbf{A}_l$ and $a_{tk}=a_{tl}$ for all $t=1, \ldots, m$. It can easily be seen that the partition π has at most $|M|\cdot V^m$ blocks. Since μ is an isomorphism, thus if $a_{tk}=a_{tl}$ $(t=1, \ldots, m)$ then the k-th component of $\mu(\delta_{\mathbf{A}}(a_t, x))$ is equal to the l-th component of $\mu(\delta_{\mathbf{A}}(a_t, x))$ for all $t=1, \ldots, m$ and $x\in X$. By this it is not difficult to prove, that the automaton $\mathbf{A}$ can be embedded isomorphically into an α_i-product of automata from the set M with at most $|M|\cdot V^m$ factors.

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References

[1] GÉCSEG, F., Composition of automata, *Proceedings of the 2nd Colloquium on Automata*, Languages and Programming, Saarbrücken, 1974, Springer Lecture Notes in Computer Science, v. 14, pp. 351—363.
[2] GÉCSEG, F., On products of abstract automata, *Acta Sci. Math.*, v. 38, 1976, pp. 21—43.
[3] GLUSKOV, V. M., Абстрактная теория автоматов, *Успехи матем. наук*, 16:5 (101), 1961, pp. 3—62.

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