Some remarks on the chromatic number of the strong product of graphs

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Let G_1, G_2 be two graphs. Let $V(G_1)=\{x_1, \ldots, x_n\}$, $V(G_2)=\{y_1, \ldots, y_m\}$ be the sets of points, $E(G_1)$, $E(G_2)$ the sets of edges. The definition of the strong product $H=G_1\times G_2$, is the following:

$$V(H) = \{(x_i, y_j) | x_i \in V(G_1),\ y_j \in V(G_2)\}$$

$$E(H) = \{((x_i, y_j), (x_k, y_l)) |\ \text{either}\ \ x_i = x_k \ \ \text{and}\ \ (y_j, y_l)\in E(G_2)$$
$$\text{or}\ \ (x_i, x_k)\in E(G_1)\ \ \text{and}\ \ y_j = y_l$$
$$\text{or}\ \ (x_i, x_k)\in E(G_1)\ \ \text{and}\ \ (y_j, y_l)\in E(G_2)\}.$$

The sets $\{(x_i, y_j)|x_i \text{ fixed}, y_j\in V(G_2)\}$ will be called *rows*, the sets $\{(x_i, y_j)|x_i\in V(G_1), y_j \text{ fixed}\}$ will be called *columns*. There are some trivial estimations for the chromatic number of the product. Let $\chi(G)$ denote the chromatic number of the graph G, we have then the following inequality (see e.g. [1], [2])

$$\max\left(\chi(G_1), \chi(G_2)\right) \leqq \chi(H) \leqq \chi(G_1)\cdot\chi(G_2).$$

The upper bound is sharp, in the sense, that we have equality in many cases, for instance if in both G_1 and G_2, the chromatic number equals to the clique number. (The clique number is the maximum cardinality of complete subgraphs.) In this sense the lower bound is not sharp. In the following we give a better lower estimation.

Let us denote by K_2 the single edge, we have then:

Theorem 1. $\chi(K_2\times G)\geqq\chi(G)+2$.

Proof. We give an indirect proof. Let $\chi(G)=k$. Let us suppose that we have coloured the product $K_2\times G$ in $k+1$ colours. $1, 2, \ldots, k+1$. In this case we can colour G as follows. Let us denote the points of K_2 by a and b. Then we can colour $x_i\in V(G)$ with the smaller one of the colours of (a, x_i) and (b, x_i) if this minimum is smaller than k. If this minimum equals k, then we colour x_i by $k-1$. This colouring is a good colouring of G. In fact, the pairs with minimum k or $k-1$ cannot be adjacent, since this would give a complete graph on 4 points, coloured in three colours, which is a contradiction. This way we have a colouring of G with $k-1$ colours and this contradicts our assumption that $\chi(G)=k$. So we have proved the theorem.

In the case when $\chi(G)=2$ our lower bound coincides with the upper bound, so this is a trivial case.

Corollary. If both G_1 and G_2 have at least one edge, then

$$\chi(G_1\times G_2) \geqq \max(\chi(G_1), \chi(G_2))+2.$$

Next we examine the case $\chi(G_1)=\chi(G_2)=3$.

Theorem 2. The product of any two odd circuits longer than 3 can be coloured with 5 colours.

Proof. Let us denote the circuit of length m by C_m. One can easily see that the colouring of $C_5\times C_5$ shown on Fig. 1 with 5 colours is a good colouring.

1	2	3	4	5
4	5	1	2	3
2	3	4	5	1
5	1	2	3	4
3	4	5	1	2

Fig. 1

For $C_5\times C_{2l+1}$ ($l>2$), we can do the following (see Fig. 2).

1 2 3	4 5						
4 5 1	2 3						
2 3 4	5 1						
5 1 2	3 4						
3 4 5	1 2						

$l-1$ times

Fig. 2

The first 5 columns are coloured in the same way as in $C_5\times C_5$, and we repeat the colouring of the 4th, and 5th columns $l-1$ times. This trivially gives a good colouring. In the case of $C_{2k+1}\times C_{2l+1}$ ($k>2$) first we colour $C_5\times C_{2l+1}$, then we repeat the colouring of the 4th and 5th rows $k-1$ times.

Remark 1. Consider the graph $K_2\times C_5$. It has 10 points. In a 5-colouration of this graph, a colour can occur at most twice, therefore each colour must occur exactly twice. So if one row contains all 5 colours then so does the other. Moreover one can easily check that if one row is coloured say (1 2 3 4 5) then the other is either (3 4 5 1 2) or (4 5 1 2 3).

Theorem 3. $C_5\times C_5$ can be coloured with 5 colours essentially uniquely.

Proof. We show, that we can colour $C_5\times C_5$ with 5 colours only so, that in every row we use all the 5 colours. Suppose indirectly that we have found a colouring, in which for instance in the first row colour 5 does not occur. Then in the second row colour 5 must occur twice by Remark 1. Continuing the colouring, in the third row we cannot have 5, in the fourth row we must find it twice, in the fifth row we cannot have number 5, which is impossible.

So suppose that the first row is coloured 1 2 3 4 5 by Remark 1 we may assume that the second row is coloured (4 5 1 2 3). The third row is therefore either (1 2 3 4 5) or (2 3 4 5 1). The first possibilty cannot occur, because the above argument applies to the columns as well, therefore the colours of the first column must be different. Going on similarly we get that the fourth and fifth rows are (5 1 2 3 4), (3 4 5 1 2).

In the sequel we present a characterization of graphs which give a five-colourable product with C_5. Before stating the theorem we need the following definition.

A homomorphism of G into H is a mapping $\varphi: V(G) \to V(H)$ for which we have that whenever $(x, y) \in E(G)$ then $(\varphi(x), \varphi(y)) \in E(H)$.

Theorem 4. Let G be a graph, for which $\chi(G) > 2$. Then $\chi(G \times C_5) = 5$ if and only if there is a homomorphism of G into C_5.

Proof. I. We know that we have a 5-colouring for $C_5 \times C_5$. Let us take a homomorphism φ of G into C_5. Let $v \in V(G)$, then $\varphi(v) \in V(C_5)$ and we colour the row $v \times C_5$ in the same way as the $\varphi(v)^{\text{th}}$ row of $C_5 \times C_5$. This colouring is obviously good because of the definition of homomorphism.

II. For the proof of the "only if" part we introduce the *5-colouration graph* of C_5. We define the *k-colouration graph* of G in the following way. Let $k \geq \chi(G)$. The vertices of the k-colouration graph are the different colourings of G with k given colours (which need not occur all in the colouring) and two vertices a, b are adjacent if and only if $G \times K_2$ can be coloured with k colours so that the colouring of the first row corresponds to a, and the colouring of the second row corresponds to b.

Lemma 1. The 5-colouration graph of C_5 has the following structure. There are 5! colouration with 5 colours in which every colour occurs exactly once. Those colourations form 4! pentagons. The remaining colourations form a bipartite graph.

The proof of this Lemma is straightforward from the proof of Theorem 3.

Continuing the proof of Theorem 4, assume that we have a 5-colouring of $G \times C_5$. Every row $v_i \times C_5$ expresses a 5-colouring of the pentagon and this 5-colouring corresponds to a point in the 5-colouration graph. We take the mapping ψ for which $\varphi(v_i)$ is this point of the 5-colouration graph, this mapping φ will be a homomorphism of G into the 5-colouration graph. Lemma 1 gives the structure of the 5-colouration graph of the pentagon and this graph has a homomorphism ψ into the pentagon (obviously). Now the mapping $\psi\varphi$ is a homomorphism of G into the pentagon, and this is what we wanted to show.

From the generalization of the above theorem, one can get the next theorem:

Theorem 5. Let $k \geq \chi(G)$, and assume that G has at least one edge. Then for a graph H we have $\chi(G \times H) \leq k$ if and only if H has a homomorphism into the k-colouration graph of G.

Proof. I. The rows of the product are copies of the graph G. So if $H\times G$ is k coloured then the colouring of a row $v\times G$ corresponds to some vertex of the k-colouration graph. This defines a homomorism of H into the k-colouration graph of G.

II. Conversely, assume that H has a homomorphism φ into the k-colouration graph of G. Then colour the row $v\times G$ as in the colouration $\varphi(v)$.

A very simple argument shows that if we take the product $K_n\times C_{2k+1}$ (K_n is the complete n-graph) then the chromatic number of this product decreases as k increases, but for $k>n$ we always get $\chi(K_n\times C_{2k+1})=2n+1$. In particular, $\chi(K_3\times C_{2k+1})=7$. One can have the feeling that if we take the product of odd circuits with 3-chromatic graphs with girth (the length of the shortest circuit in the graph) larger than 3, we can get smaller chromatic number for the product, probably only 5. We show some examples which contradict to this tendency.

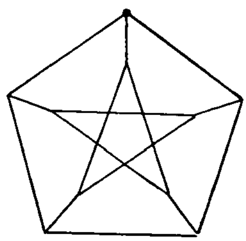

Fig. 3

Lemma 2. Let $\alpha(G)$ denote the maximum number of independent points in G. Then for any G and k, we have the inequality:

$$2\alpha(G\times C_{2k+1}) \leqq (2k+1)\alpha(G).$$

Proof. The maximum number of independent points in $G\times K_2$ is the same as in G. Summing this for all edges of C_{2k+1} the inequality follows.

Let P_5 denote the Petersen-graph shown in Fig. 3. It is obvious that $\chi(P_5)=3$. Then we have the following theorem.

Theorem 6. The chromatic number of the product $P_5\times C_{2l+1}$ is always greater than 5.

Proof. We have $\alpha(P_5)=4$. From Lemma 2 we get the inequality:

$$\alpha(P_5\times C_{2l+1}) \leqq 2(2l+1).$$

We show that here the equality cannot hold. Suppose it does. Then equality holds in the proof of Lemma 2 on all edges of C_{2l+1}.

Let us consider the independent vertices in the first row of the product $P_5\times C_{2l+1}$. Let its number be f ($f=0, 1, 2, 3, 4$). We can choose from the second row at most $4-f$ independent vertices. Since equality holds in the proof of Lemma 2, we have precisely $4-f$ vertices from the second row. From the third row we must choose f independent vertices again. Continuing this procedure from the $(2l)^{\text{th}}$ row we must choose $4-f$ vertices. From the $(2l+1)^{\text{st}}$ row we must choose f vertices because this row is the neighbour of the $(2l)^{\text{th}}$ row, but this is the neighbour of the first row too, se we must choose $4-f$ vertices from the $(2l+1)^{\text{st}}$ row. This is possible only if $f=4-f$, $f=2$. It can be easily seen that if we take in P_5 two independent vertices, then they uniquely determine the maximal independent set which contains these vertices. In the case $f=2$ this gives that in every $(2i+1)^{\text{st}}$ row ($i<l$) we have the same two vertices. But this excludes any vertices in the $(2l+1)^{\text{st}}$ row because it is the neighbour of both of the first and $(2l)^{\text{th}}$ row. Thus we get

$$\alpha(P_5\times C_{2l+1}) < 2(2l+1).$$

From this and from the well-known inequality

$$\frac{|V(G)|}{\alpha(G)} \leqq \chi(G). \tag{1}$$

We get that $\chi(P_5 \times C_{2l+1}) > 5$ for any l.

Let us take the "generalized Petersen-graph", P_{13}, seen in Fig. 4.

One can easily see that $\chi(P_{13})=3$. The length of the shortest circuit is 7. For this graph we have the following theorem:

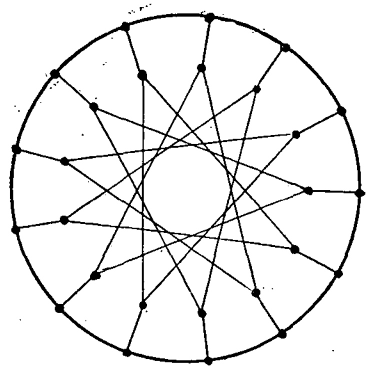

Fig. 4

Theorem 7. For any k $\chi(P_{13} \times C_{2k+1}) > 5$.

Proof. Let us consider the maximal number of independent vertices in P_{13}. In the outer and the inner circuit there can be at most six independent vertices. We show that if we have in the outer circuit six independent vertices, then we can have in the inner circuit at most four vertices. The outer six independent vertices exclude their six inner neighbours (seei Fg. 5a).

It is essentially unique to choose six independent vertices in the outer circuit. The remaining seven vertices in the inner circuit consist of one isolated vertex and from three independent edges and from this graph we can choose at most four independent vertices (as it is indicated in Fig. 5a).

We can argue similarly in the case when we have six independent vertices in the inner circuit (see Fig. 5b). Thus $\alpha(P_{13})=10$. Using Lemma 2 we have the following inequality:

$$\alpha(P_{13} \times C_{2k+1}) \leqq (2k+1) \cdot 5.$$

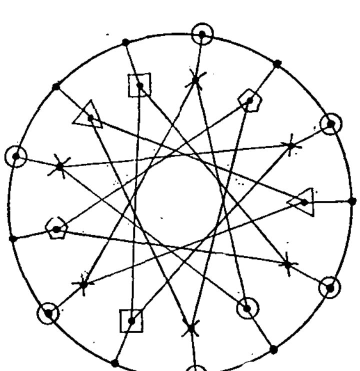

Fig. 5a

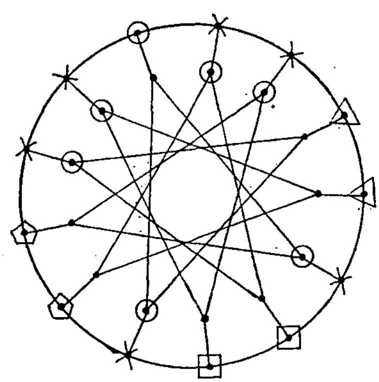

Fig. 5b

From this and using inequality (1) we have:

$$\chi(P_{13}\times C_{2k+1}) \geqq \frac{2\cdot 13\cdot(2k+1)}{5(2k+1)} > 5.$$

So we have proved the theorem.

Problems. 1. Give a better lower bound for the $\chi(G_1\times G_2)$ if $\chi(G_1)$, $\chi(G_2)$ are larger than 3.

2. Prove that for any large g one can find G_1 and G_2, for which $\chi(G_1)=\chi(G_2)=3$, the girth of both graphs is larger than g but $\chi(G_1\times G_2)\geqq 6$.

3. It would be interesting to determine the structure of the k-colouration graphs of some classes of graphs, to get similair results as in Theorem 4.

Reference

[1] BERGE, C., *Graphs and hypergraphs*, North Holland, 1973.
[2] BOROWIECKI, M., On chromatic number of products of two graphs, *Colloq. Math.*, v. 25, 1972, pp. 49—52.

(Received April 12, 1978)