Basic theoretical treatment of fuzzy connectives

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Introduction

One of the most interesting problems in the theory of fuzzy sets is the choice of the fuzzy connective operations, i.e. the union and the intersection.

Definition 1. The fuzzy set μ is an arbitrary function

$$\mu: X \to [0, 1] \tag{1}$$

interpreted on the non-empty universal discourse X.

In such a sense, the characteristic function of the common sets is a special fuzzy set.

Zadeh (1965) [24] extended the intersection and union of the subsets of the common sets in the following way

$$\mu_{A\cup B}(x) = \max(\mu_A(x),\ \mu_B(x)) \quad \text{for all} \quad x \in X \quad \text{and}$$

$$\mu_{A\cap B}(x) = \min(\mu_A(x),\ \mu_B(x)) \quad \text{for all} \quad x \in X, \tag{2}$$

where $\mu_{A\cup B}$ and $\mu_{A\cap B}$ are the fuzzy sets corresponding to $A \cup B$ and $A \cap B$, respectively.

Below we shall survey in broad outlines the development of the views relating to fuzzy operations. Historical survey of fuzzy operations:

Besides operations (1), others also have been proposed for the generalization of the operations in set theory [24], [17]. Some examples are

$$\mu_{A\cap B}(x) = \mu_A(x) \cdot \mu_B(x) \quad \text{and}$$

$$\mu_{A\cup B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x) \tag{3}$$

or

$$\mu_{A\cap B}(x) = \max(\mu_A(x) + \mu_B(x) - 1, 0) \quad \text{and}$$

$$\mu_{A\cup B}(x) = \min(\mu_A(x) + \mu_B(x),\ 1). \tag{4}$$

All this reveals the arbitrary nature of the definitions. This arbitrariness can be resolved with a basis on the axiom system general in mathematics. Strivings in this direction were first made in defence of the min and max operations [3], [12], [9].

In effect, this merely involved the characterization of operations (1) with other properties.

Subsequently, other axiom systems were created [11], [12], [14], which were not represented by operations (1); there were publications in which algebraic structures were investigated without representation [2], [13], [15]. Here the emphasis was on the rational establishment of the axioms.

A whole series of axiomatic examinations arose for the most varied operations; however, these were unable to unify the views relating to the operations, but rather made the problem more ramified. Study of the mutual interrelations between the axiom systems might have led to a solution, but very great difficulty was caused by the fact that it was impossible to compare the axioms. Only one such study has been made [10].

One possibility was to return to the bases, i.e. to base the rational nature of the axioms not on opinions, but on empirical examinations. The first such examination did not relate exactly to this, but to the question of whether the created operations correspond to practice [21]. The result was that they do not.

Further, it is not advisable to make a mathematical theory dependent on narrow empirical examinations; rather, operation classes must be produced from which the appropriate operation can be selected in a manner adequate to the practical requirements.

The operations should if possible be made flexible. Parameter-dependent operation series were produced by Yager [23] and by Hamacher [11], but these were as individual as the earlier operations. Although operation classes were defined, a practical interpretation of the parameters did not materialize.

The next period was characterized by the appearance of monographs on operations and axiom systems [6], [22].

These works ensured a possibility for the discovery of the common properties of operations and axiom systems and for the selection of a minimal axiom system [4]. However, only a narrow range of the examined operations could be characterized with these axiom systems.

The axioms of this minimal system are the strict monotonity of the operations, the holding of the correspondence principle, associativity and continuity. The adoption of these axioms can be based rationally in the following way:

The correspondence principle is satisfied by all fuzzy operations, i.e. their restriction to the characteristic function is a classical set-theory operation. The associativity holds for every operation examined so far, and in addition a possibility is created for the extension of two-variant operations to multi-variant ones. The lack of continuity terminates the homogeneous effect of the operation.

Strict monotonity is not satisfied by every operation; its condition rather served the realization of the representation. However, the condition of monotonity exists for all operations.

Thus, it is advisable to carry out an examination of not strictly monotonous operations. Hence, we must obtain, for example, (4) and (1).

The main result in the paper is the giving of representations of all operations of such type, as functions of various conditions.

The study relies on the theory of ordered semigroups [8], [20] and the associative function equations [1].

1. Fuzzy algebra

Let I be the closed interval $[0, 1]$ of the real numbers. This notation partly serves to simplify the description, and partly refers to the generalizability of the theorems and definitions.

The set of all the fuzzy sets (1) is

$$F(X) = \{\mu | \mu \colon X \to I\} \tag{5}$$

(we shall denote $F(X)$ briefly by F). Let Ch be a set of common characteristic functions.

Definition 2. The fuzzy sets μ and ν are said to be equal if

$$\mu(x) = \nu(x) \quad \text{for all} \quad x \in X. \tag{6}$$

The fuzzy set μ precedes the fuzzy set ν if

$$\mu(x) \leqq \nu(x) \quad \text{for all} \quad x \in X. \tag{7}$$

Theorem 1. The relation $\leqq$ is a partial ordering on F. Let us consider an n-ary algebraic operation

$$* \colon F^n \to F \quad (n = 1, 2, \ldots) \tag{8}$$

on the set F of fuzzy sets.

Definition 3. The operation $*$ is isotonic (antitonic) if it follows from the inequalities

$$\mu_i \leqq \nu_i \quad (i = 1, 2, \ldots, n)$$

that

$$\mu_1 * \ldots * \mu_n \leqq \nu_1 * \ldots * \nu_n \quad (\mu_1 * \ldots * \mu_n \geqq \nu_1 * \ldots * \nu_n) \tag{9}$$

for all $(\mu_1, \ldots, \mu_n), (\nu_1, \ldots, \nu_n) \in F^n$. The isotonic and antitonic operations together are said to be monotonic.

The ordering relation $\leqq$ interpreted on the fuzzy sets is a generalization of the partial ordering defined by the entailment interpreted on the common sets.

Definition 4. By fuzzy algebra [5] (the algebra of fuzzy sets) we understand all those algebraic structures interpreted on F for which it holds that

(A1) all of its operations are monotonic.

Fuzzy algebra is said to be "ordinary" if the following condition also holds:

(A2) the restriction of all of its algebraic operations to Ch agrees with some set-theory operation with the same number of variables.

In our work we shall examine those ordinary fuzzy algebras $\langle F, * \rangle$ (in the following simply fuzzy algebra) which satisfy the following conditions:

(F1) $*$ is a binary connective operation, i.e. its restriction to Ch is either intersection interpreted on the normal sets, or union.

Let us consider those fuzzy algebraic operations for which there is a function $f \colon I \times I \to I$ such that

$$(\mu * \nu)(x) = f(\mu(x), \nu(x)) \quad \text{for all} \quad x \in X. \tag{10}$$

The attribution $* \to f$ is mutually unambiguous. Let us denote the set of fuzzy algebraic operations with this property by Z.

(F2) Let $*$ be an operation belonging to Z.

Theorem 2. Let $*: F\times F\to F$ be an operation in Z. The algebraic structure $\langle F, *\rangle$ satisfying condition F1 is fuzzy algebra if, and only if, it holds for the function f ascribed to $*$ that

(i) f is monotonic in the sense agreeing with $*$;

(ii) $f(0, 0)=0, f(1, 1)=1$, and further, if the restriction of the operation $*$ to Ch is intersection (union); then $f(0, 1)=f(1, 0)=0$ $(f(0, 1)=f(1, 0)=1)$.

Proof. Let $\langle F, *\rangle$ be the fuzzy algebra satisfying condition F1.

(i) Let us assume that $*$ is isotonic. Let $x_1, x_2, y_1, y_2\in I$, so that $x_1\leqq x_2$ and $y_1\leqq y_2$. Let us consider the fuzzy sets

$$\mu_1(x) = x_1,\ \mu_2(x) = x_2,\ \nu_1(x) = y_1,\ \nu_2(x) = y_2 \quad \text{for every} \quad x\in X.$$

For these it holds that

$$\mu_1 \leqq \mu_2 \quad \text{and} \quad \nu_1 \leqq \nu_2.$$

It follows from the isotonity of operation $*$ that

$$\mu_1 * \nu_1 \leqq \mu_2 * \nu_2.$$

Taking F2 into consideration:

$$f(\mu_1(x),\ \nu_1(x)) \leqq f(\mu_2(x),\ \nu_2(x)) \quad \text{for all} \quad x\in X.$$

It therefore follows from $x_1\leqq x_2$ and $y_1\leqq y_2$ that

$$f(x_1, y_1) \leqq f(x_2, y_2),$$

i.e. f is isotonic. The postulate can be demonstrated similarly for the antitonic case.

(ii) The postulate arises simply from consideration of A2 or F1 and F2.

Proof of the inverse of the postulate is likewise simple.

Consequence: with the operation f ascribed to $*$ I is an ordered algebraic structure.

Theorem 2 ensures that study of the representations of the algebraic structure determined by the operation f ascribed to the operation $*$ is sufficient for examination of the representations of the fuzzy algebras $\langle F, *\rangle$ satisfying conditions F1 and F2.

As concerns f, let us assume that

(F3) f is associative;

(F4) f is continuous on $I\times I$.

It can readily be seen that the operation $*$ determined by such f is associative and continuous from point to point, i.e. if the series of fuzzy sets $\{\mu_n\}$ and $\{\nu_n\}$ converge from point to point to the fuzzy sets μ and ν, then the series of fuzzy sets $\{\mu_n * \nu_n\}$ converges from point to point to the fuzzy set $\mu * \nu$.

In the following section postulates will be given for the case when the restriction to Ch of the operation $*$ determined by f is the normal set-theory intersection. In this case we denote the determining function by c. The function corresponding to the union is denoted by d. The postulates for c and their proofs can be applied appropriately to d.

2. Representation theorem

Let us first summarize the properties having by the function $c: I\times I\to I$ defined in section 2.

(T1) c is monotonous;
(T2) $c(0, 0)=0$, $c(1, 1)=1$, $c(0, 1)=c(1, 0)=0$;
(T3) c is associative;
(T4) c is continuous.

Theorem 3. If T1 and T3 hold for c, then
(T1′) c is isotonic [8].
Thus, the set I forms a semigroup completely ordered with operation c.

Definition 5. The function h is said to be Archimedean in the interval $[a, b]$ if

$$h(x, x)<x \quad \text{for all} \quad x\in(a, b). \tag{11}$$

The representation theorem relating to the Archimedean case was proved by Ling [16] by means of elementary analysis. The theorem can be derived from the earlier result of Mostert and Shields [18].

We shall make use of this theorem in the following.

Theorem 4. Let J be a closed interval $[a, b]$ of real numbers, and h the function $h: J\times J\to J$. h has the properties that
(i) h is monotonous;
(ii) h is associative;
(iii) h is continuous;
(iv) $h(a, a)=a$, $h(b, b)=b$, $h(b, x)=h(x, b)=x$ $(x\in X)$; (12)
(v) h is Archimedean
if and only if there exists a continuous, strictly monotonously decreasing function g, mapping the interval $[a, b]$ into the interval $[0, \infty]$ for which $g(b)=0$ such that h may be represented in the form

$$h(x, y) = g^{(-1)}(g(x)+g(y)) \tag{13}$$

where $g^{(-1)}$ is the pseudo-inverse of g

$$g^{(-1)}(x) = \begin{cases} g^{-1}(x) & \text{if} \quad g(b)\leqq x\leqq g(a), \\ a & \text{if} \quad g(a)\leqq x, \end{cases} \tag{14}$$

where g^{-1} is the normal inverse of function g in $[g(b), g(a)]$.

Function g is termed the additive generator of the Archimedean operation h, and g is unambiguously determined apart from a positive constant, i.e. $\alpha\cdot g$ $(\alpha>0)$ likewise generates h.

It should be noted that the theorem can also be stated in such a way that the generator function g' maps the interval $[a, b]$ into $[-\infty, 0]$, it increases strictly monotonously, and $g(b)=0$. In this case the definition of the pseudo-inverse is modified appropriately.

Function c with properties T1—T4 satisfies conditions (i)—(iv) of Theorem 4. In the following we shall not restrict our considerations to the Archimedean case. Mostert and Shields have carried out similar examinations relating to semigroups [18].

Let us consider the set of the idempotent points of the interval I

$$N = \{x \mid x \in I, \quad c(x, x) = x\}. \tag{15}$$

Theorem 5. N is a closed set.

Proof. We see that N contains every accumulation point. Let x_0 be an optional accumulation point of N. A point series x_n may then be selected from N such that

$$\lim_{n \to \infty} x_n = x_0.$$

Since $x_n \in N$ for every n, we have $c(x_n, x_n) = x_0$, and c is continuous (T4), so that

$$c(x_0, x_0) = \lim_{x_n \to x_0} c(x_n, x_n) = \lim_{x_n \to x_0} x_n = x_0,$$

and thus $x_0 \in N$.

Let $M = I \setminus N$. M is a restricted and open point set. Let us assume that M is not empty.

Theorem 6. M can be constructed as the combination of a finite or infinitely large number of open intervals, not projecting into one another in pairs, the end-points of which do not belong to M [21].

Therefore M has the form

$$M = \bigcup_{i \in P} M_i \tag{16}$$

where P is a finite or an infinite index set and $M_i = (a_i, b_i)$, for which, if $x \in (a_i, b_i)$,

$$c(x, x) \neq x, \tag{17}$$

while $c(a_i, a_i) = a_i$ and $c(b_i, b_i) = b_i$.

Let us select an optional region $[a_i, b_i] \times [a_i, b_i]$. In this region it holds too that c is isotonic (T1′), associative (T3) and continuous (T4). For determination of the properties corresponding to T2, let us consider the following theorems:

Theorem 7. For every $x \in [a_i, b_i]$:

(i) $$c(a_i, x) = c(x, a_i) = a_i, \tag{18}$$

(ii) $$c(b_i, x) = c(x, b_i) = x. \tag{19}$$

Proof. First, we see that

$$c(1, x) = c(x, 1) = x \quad \text{for all} \quad x \in I. \tag{20}$$

On the basis of (T2), $c(0, 1) = 0$ and $c(1, 1) = 1$, and with consideration of the continuity (T4) the function $c(x, 1)$ therefore maps I on I. Then, for any $y \in I$ there exists an $x \in I$ such that $c(x, 1) = y$. Utilizing this fact and the associativity (T3).

$$c(y, 1) = c(c(x, 1), 1) = c(x, c(1, 1)) = c(x, 1) = y \quad \text{for all} \quad y \in I.$$

Part (i) of the theorem is a simple consequence of the isotonity (T1′) and (20)

$$a_i = c(a_i, a_i) \leqq c(x, a_i) \leqq c(1, a_i) = a_i.$$

The proof of part (ii) is the application of that of (20) to $[a_i, b_i]$.

Theorem 8. For every $x\in(a_i, b_i)$, $c(x, x)<x$.

Proof. As a consequence of the isotonity (T1′) and (19)

$$c(x, x) \leqq c(b_i, x) = x \quad \text{for all} \quad x\in[a_i, b_i].$$

If $x\in(a_i, b_i)$, then $c(x, x)\neq x$, so that

$$c(x, x) < x \quad \text{for all} \quad x\in (a_i, b_i).$$

Theorem 9. For every $(x, y)\in[a_i, b_i]\times[a_i, b_i]$

$$c(x, y) \leqq \min (x, y). \tag{21}$$

Proof. As a consequence of (19)

$$c(x, y) \leqq c(x, b_i) = x \quad \text{and} \quad c(x, y) \leqq c(b_i, y) = y,$$

therefore,

$$c(x, y) \leqq \min (x, y).$$

Theorem 10. Let $H=I^2\setminus \bigcup_{i\in P} M_i^2$. Then

$$c(x, y) = \min (x, y) \quad \text{for all} \quad (x, y)\in H. \tag{22}$$

Proof. Let us assume that $x\leqq y$. Let $(x, y)\in H$.
(i) If $x\in N$ and $y\in I$, then

$$x = c(x, x) \leqq c(x, y) \leqq c(x, 1) = x.$$

(ii) If $x\in(a_i, b_i)\subseteqq M$ and $y\notin (a_i, b_i)$, then

$$x = c(x, b_i) \leqq c(x, y) \leqq c(x, 1) = x.$$

In both cases $c(x, y)=x=\min (x, y)$.

Let c_i be the restriction of the function c to the region $[a_i, b_i]\times[a_i, b_i]$. As a consequence of the equalities $c(a_i, a_i)=a_i$ and $c(b_i, b_i)=b_i$ as well as the isotonity (T1′) and continuity (T4) of c, c_i maps the region $[a_i, b_i]\times[a_i, b_i]$ on $[a_i, b_i]$.

To summarize, c_i satisfies conditions T1′, T3, T4 and T2′

(T2′) $$c_i(a_i, a_i) = a_i, \; c_i(b_i, b_i) = b_i,$$
$$c_i(a_i, b_i) = c_i(b_i, a_i) = a_i$$

and by Theorem 8 it is Archimedean. From the Ling theorem, therefore, for every $i\in P$ there exists a generator function g_i additive in $[a_i, b_i]$ to c_i.

Thus, the following theorem holds for c:

Theorem 11. Let c be the function $c\colon I\times I\to I$. c satisfies conditions T1—T4 if and only if c has the form

$$c(x, y) = \begin{cases} g_i^{(-1)}(g_i(x)+g_i(y)), & \text{if } (x, y)\in M_i^2 = (a_i, b_i)^2 \;\; i\in P \\ \min (x, y), & \text{if } (x, y)\in I^2\setminus \bigcup_{i\in P} M_i^2 \end{cases} \tag{23}$$

where $\{M_i\}_{i\in P}$ is the sum of a finite or infinitely large number of open intervals belonging to I, not projecting into one another in pairs. g_i is a function mapping the closed interval $[a_i, b_i]$ into the interval $[0, \infty]$, which is a continuous, strictly monotonously decreasing function, and $g_i(b_i)=0$. $g^{(-1)}$ is the pseudo-inverse of g_i. (It should be noted that P may be empty.)

Proof. (i) Let us assume that T1—T4 hold for the function $c: I\times I\to I$. If every point of I is idempotent, i.e. $I=N$, then on the basis of Theorem 10, $c(x, y)= =\min(x, y)$ for all $(x, y)\in I^2$. If $N\subset I$, then as a consequence of Theorems 5 and 6 in I^2 there are regions $[a_i, b_i]\times[a_i, b_i]$ not projecting into one another in pairs, in which the functions c_i satisfy the conditions of Theorem 4 (Ling) by Theorems 7—10. In the given region, therefore, there exist generator functions g_i additive for c_i-s. Outside such regions, from Theorem 10: $c(x, y)=\min(x, y)$.

(ii) Let us assume that the function $c: I\times I\to I$ exists in the form (23). If P is empty, then $c(x, y)=\min(x, y)$ for all $(x, y)\in I^2$. Therefore T1—T4 hold.

If P is not empty, then by Theorem 4 (Ling) the function c is isotonic, associative and continuous separately both in the regions $\{M_i^2\}$ $(i\in P)$ and outside these regions.

Because of (12), at the limit of the regions M_i^2, $c(x, y)=\min(x, y)$, and c therefore has no breakpoint. Thus, c is continuous (T4) in I^2. T2 similarly follows from these arguments.

The proof of the isotonity (T1′) and the associativity (T3) is lengthy, and accordingly we do not present it here.

Without proof, we list some of the consequences of Theorem 11.

Theorem 12. Every function $c: I\times I\to I$ satisfying conditions T1—T4 is commutative.

Definition 6. The function $t: I\times I\to I$ is said to be a t norm [19] if

(i) $t(0, 0)=0$, $t(x, 1)=t(1, x)=x$ for all $x\in I$,

(ii) t is isotonic,

(iii) t is commutative, and

(iv) t is associative.

Definition 7. The function $t: I\times I\to I$ is said to be a strict t norm if (i) and (iv) hold, and

(v) t is continuous, and

(vi) t is strictly isotonic, i.e.

$$t(x_1, y) < t(x_2, y) \quad \text{if} \quad 0 < x_1 < x_2 \leqq 1,$$

$$t(x, y_1) < t(x, y_2) \quad \text{if} \quad 0 < y_1 < y_2 \leqq 1.$$

Theorem 13. Every function $c: I\times I\to I$ satisfying conditions T1—T4 is a continuous t norm.

If we assume strict monotonity instead of T1 for function c, then it is a strict t norm and Archimedean in I.

Studies relating to continuous t norms have been performed by Schweizer and Sklar [19], [20].

Finally, let us examine the possibility of constructing the $\min(x, y)$ function by means of a generator function. By Theorem 4 (Ling) there is no additive generator of form (13), as it is not Archimedean. Ling studied this problem in some detail [16].

Theorem 14. Let J be the closed interval $[a, b]$ of the real number straight line. If $c(x, y) = \min(x, y)$ for all $(x, y) \in [a, b] \times [a, b]$, then there does not exist a continuous function $g: [a, b] \to [0, \infty]$ such that c can be represented in the form

$$\min(x, y) = g^*(g(x) + g(y)) \tag{24}$$

where it holds for (the not unconditionally continuous) g^* that $g^*(g(x)) = x$ for all $x \in [a, b]$.

Theorem 15. Assume that J and c satisfy the conditions of Theorem 14. Then, there does not exist a strictly monotonously decreasing function $g: [a, b] \to [0, \infty]$ such that c can be represented in the form

$$\min(x, y) = g^*(g(x) + g(y))$$

where g^* is the function defined in Theorem 14.

A connection may be created between the generator functions and $\min(x, y)$ from another aspect. Let $g(x)$ be the additive generator function of $c(x, y)$.

Theorem 16. $g^\lambda(x)$ $(\lambda > 0)$ also has the properties of the generator functions.

Theorem 17. If $c_\lambda(x, y)$ is an operation determined by the generator function $g^\lambda(x)$, then

$$\lim_{\lambda \to \infty} c_\lambda(x, y) = \min(x, y).$$

Theorems 16 and 17 for strictly monotonous functions $c(x, y)$ have been proved by Dombi [4].

3. Examples

(i) Zadeh [24]

$$c(x, y) = \min(x, y),$$
$$(c(x, x) = x, x \in I).$$

(ii) Lukasievicz [17]

$$c(x, y) = \max(x + y - 1, 0)$$
$$g(x) = \begin{cases} 1 - x, & \text{if } x \leqq 1, \\ 0, & \text{if } x > 1 \end{cases}$$

(not strictly monotonous, Archimedean).

(iii) [24]

$$c(x, y) = x \cdot y,$$
$$g(x) = -\log x$$

(strictly monotonous).

(iv) Dubois [7]

$$c(x, y) = \frac{x \cdot y}{\max(x, y, \lambda)} = \begin{cases} \dfrac{x \cdot y}{\lambda} & \text{if } \lambda > x, y, \\ \min(x, y), & \text{otherwise,} \end{cases}$$

$$g(x) = -\log \frac{x}{\lambda}, \quad \text{if } x > 0,$$

(v) Hamacher [12]

$$c(x, y) = \frac{\lambda \cdot x \cdot y}{1-(1-\lambda)\cdot(x+y-x\cdot y)},$$

$$g(x) = -\log \frac{\lambda \cdot x}{1+(\lambda-1)\cdot x}.$$

(vi) Yager [23]

$$c(x, y) = \begin{cases} 1-((1-x)^\lambda+(1-y)^\lambda)^{1/\lambda}, & \text{if } (1-x)^\lambda+(1-y)^\lambda < 1, \\ 0, & \text{otherwise,} \end{cases}$$

$$g(x) = \begin{cases} (1-x)^\lambda, & \text{if } x < 1, \\ 0, & \text{if } x \geqq 1. \end{cases}$$

(vii) Dombi [4]

$$c(x, y) = \frac{1}{1+\left(\left(\frac{1}{x}-1\right)^\lambda+\left(\frac{1}{y}-1\right)^\lambda\right)^{1/\lambda}},$$

$$g(x) = \left(\frac{1}{x}-1\right)^\lambda$$

4. Conclusion

The objective outlined in the Introduction has been attained. The square resolution existing in the general case is based on the non-Archimedean nature. If we do not desire such a resolution, then the operations must be restricted to the Archimedean case.

Modification of other conditions means the possibility of a further step in the investigations. An example is the study of the non-continuous case, e.g.

$$t(x, y) = \begin{cases} x, & \text{if } y = 1, \\ y, & \text{if } x = 1, \\ 0, & \text{if } x \neq 1 \text{ and } y \neq 1, \end{cases}$$

which otherwise satisfies T1—T3.

Setting out from the generator functions, another research area is the characterization of the possible operation classes, or the study of the connection between various operations, e.g. generalization of the DeMorgan laws.

The question still remains of what connection exists between the empirical examinations and the fuzzy algebraic operations. The research up to date has not provided a satisfactory answer to this.

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