On certain partitions of finite directed graphs and of finite automata

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I. Introduction and basic terminology

§ 1.

The main aim of this paper is to study the partitions π of the vertex set of a finite directed graph G such that π satisfies the following condition: if the vertices a, b are in a common class modulo π and the edges $\overrightarrow{ac}$, $\overrightarrow{bd}$ exist in G, then c, d are also in a common class. These partitions will be called partitions having property P in the paper.

My attention was called to studying these partitions by the automaton-theoretical articles [7], [9], [10]. The majority of the present paper is written, however, from a graph-theoretical point of view.

Sections 3—5 are devoted to introducing the notions which are basic for the paper, and to exposing a few simple consequences of the definitions.

In Chapter II a description of the P-partitions of functional graphs will be given. The results of Chapter II will be generalized in Chapter III into an overview of the P-partitions of all (finite, directed) connected graphs in which no vertices with out-degree zero occur.

Chapter IV contains comments of several types. The extension of the former results to non-connected graphs is sketched, their extension to graphs with sinks is questioned and examples answering some arising questions will be given. § 12 is an appendix to the paper; it starts with lemmas on a sequence of partitions of the state set of a Moore automaton, later these facts lead to a proof solving a problem[1] on the complexity and state number of Moore automata.

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[1] Conjecture 1 in [4].

§ 2.

The idea of studying the P-partitions by graph-theoretic methods was suggested by the articles [7], [9], [10] of Gill, Flexer and Hwang. They have dealt with questions concerning automata. The mentioned partition type is the same as the "partitions with substitution property" in their papers.[2]

Gill, Flexer and Hwang discussed mainly the partitions of the state set of an automaton such that the factor automaton (modulo the partition in question) exists and is a cycle. It turns out from their articles that the overview of these partitions has a certain technological significance.[3]

Yoeli and Ginzburg [14] introduced the P-partitions under the name "admissible partitions". They investigated chiefly the atoms[4] in the lattice of these partitions.

Dvořák, Gerbrich and Novotný deal in their most recent paper [5], essentially, with connected directed graphs in which no out-degree exceeds one. They describe all the possible homomorphisms (if there exists any) of a graph onto another.

The question, to whose solution Chapter II of the present paper is devoted, is the same (apart from terminological differences) as the problem of describing the congruences of connected finite unary algebras with one operation. The papers [11], [15], [16], [17] of Kopeček, Egorova and Skornjakov deal with somewhat related questions.

The autonomus semiautomata (possibly infinite ones) which are investigated by Machner and Strassner in [12] are essentially the same as the functional graphs in our terminology. Theorem 3 and Corollary 6′ in [12] concern to finite functional graphs, these results correspond to certain considerations in our Chapter II. The mentioned results are derived by Machner and Strassner as consequences of their investigations dealing with the infinite case.

§ 3.

By a graph, we mean a connected directed finite graph. Parallel edges with the same orientation are not permitted. We allow, however, loops and oppositely oriented parallel edges. Sometimes we regard a graph G as a relational structure, this means that we say "the relation $\alpha_G(a, b)$ holds" instead of saying "the edge from the vertex a to the vertex b exists in G".[5]

The most familiar notions of the theory of directed graphs are supposed to be known; especially, the notions of *path* and *cycle*. These are understood always in directed sense, and with pairwise different vertices. (Of course, the first vertex of a cycle and the last one are the same.)

The notion of *circuit* originates from the notion of cycle by the modification that the edges are considered as non-directed ones.

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[2] In [7], [9], [10] automata without output signs are considered. Actually, the graph of an automaton is studied rather than the graph itself.

[3] See the middle of Section 1 in [7] and Section VII of [9].

[4] π is called an atom if $\pi \supset o$ holds and $\pi \supseteqq \pi' \supset o$ implies $\pi = \pi'$ (where π, π' are P-partitions).

[5] The subscript G is possibly dropped in α_G if its absence cannot cause a misunderstanding. Similar notational simplifications may occur in other cases, too.

A vertex with in-degree zero is called a *source*. A vertex with out-degree zero is called a *sink*.

The lattice of all partitions of the vertex set V of a graph G is denoted by $L(V)$; as usual, $\pi_1 \subseteq \pi_2$ means that $\pi_1(\in L(V))$ is a (proper or non-proper) refinement of $\pi_2(\in L(V))$. ι is the partition having one class only, and o is the partition each class of which consists of a single element.

Consider a partition $\pi(\in L(V))$. We say that π *possesses the property* P (or, simply, that π is a P-partition) if

$$(a \equiv b \,(\text{mod}\, \pi) \,\&\, \alpha(a, c) \,\&\, \alpha(b, d)) \Rightarrow c \equiv d \,(\text{mod}\, \pi).$$

holds universally (i.e., for every choice of the vertices a, b, c, d).

Denote by $[a]_\pi$ (or simply by $[a]$) the class (modulo π) containing a vertex a. The *factor graph* $G^* = G/\pi$ is defined in the following manner:

the vertices of G^* are the classes of V modulo π,

$\alpha^*([a], [b])$ holds[6] if and only if there exist two vertices $a'(\in V), b'(\in V)$ such that $a' \in [a], b' \in [b]$ and $\alpha(a', b')$.

It is clear that G/π can have loops even if G is loop-free.

We end this § by asserting two obvious statements concerning the above notion of the factor graph. The first of them is an analogon of one of the general isomorphism theorems of universal algebras.

Lemma 1. *Let π_1, π_2 be two partitions of the vertex set V of a graph G. Suppose $\pi_1 \subseteq \pi_2$; denote by π_2' the following partition of the vertex set of G/π_1: $[a]_{\pi_1} \equiv [b]_{\pi_1}$ (mod π_2') if and only if $a \equiv b$ (mod π_2). Then G/π_2 and $(G/\pi_1)/\pi_2'$ are isomorphic.*

Lemma 2. *Let π be a partition of the vertex set of a graph G. If there is no source (or no sink) in G, then there is no source (or no sink, resp.) in G/π, too.*

§ 4.

A graph G is called a *functional graph* if the out-degree of each vertex of G is one. A simple structural description of the finite functional graphs is due to Ore (see [13], § 4.4; [1], Chapter I); his theorem states that *a connected graph G is functional if and only if*

G has precisely one circuit,

the circuit in G is a cycle, and

each other edge of G is directed towards the cycle.

By its definition, a functional graph G does not contain a sink; G contains at least one source unless G is a cycle.

The vertices and edges of the cycle of a functional graph G are called *cyclic*. Each other vertex and edge of G is said *acyclic*. (A source is always acyclic.)

If a is a vertex of a functional graph G, then we denote by $\varphi_G(a)$ the (uniquely determined) vertex b for which $a_G(a, b)$ is true. We define $\varphi^i(a)$ by $\varphi^i(a) = \varphi(\varphi^{i-1}(a))$ recursively; we agree that $\varphi^0(a) = a$.

[6] We write α^* instead of α_{G^*}.

Let a, b be two vertices of a functional graph. If there is a number $i(\geqq 0)$ such that $\varphi^i(a)=b$, then we denote by $\chi(a, b)$ the smallest of these numbers.

Let a path in a functional graph G be considered whose vertices are

$$a_1, a_2, \ldots, a_s \quad (s \geqq 2). \tag{4.1}$$

If $a_1, a_2, \ldots, a_{s-1}$ are acyclic vertices and a_s is cyclic, then we call (4.1) a *principal path*. To each acyclic vertex a_1, there is exactly one principal path starting from a_1.

Let B be a subset of V such that every element of B is an acyclic vertex. B is called a *basic set* if, to each $b(\in B)$, the principal path starting with b contains no other element of B than b. The empty set is regarded to be basic, too. (Thus each functional graph — even a cycle — has at least one basic set.) The set of all sources of a functional graph is always basic. For any basic set B, a principal path may contain at most one element of B.

Consider a basic set B of a functional graph. A vertex a is called *outer with respect to* B if a is acyclic and the principal path starting with a contains an element of B. The remaining vertices are called *inner (with respect to B)*. The following lemma is obvious:

Lemma 3. *Let B be a basic set in a functional graph. Then*
(i) *each element of B is outer with respect to B,*
(ii) *each cyclic vertex of the graph is inner with respect to B, and*
(iii) *if a is inner with respect to B, then $\varphi(a)$ is also inner.*

In the last assertion of this § we state a connection between the P-partitions and a slight extension of the class of functional graphs.

Proposition 1. *Let G be a graph and π be a partition of its vertex set. π has the property* P *if and only if each out-degree in the factor graph G/π is either zero or one.*

Proof. The out-degree of a vertex $[a]$ of G/π is at least two if and only if there exist four vertices b_1, b_2, c, d in G such that $b_1 \in [a]$, $b_2 \in [a]$, $\alpha(b_1, c)$, $\alpha(b_2, d)$ and $c \not\equiv d \pmod{\pi}$. This condition is precisely the negation of the property P.

§ 5.

In this last section of Chapter I, a few concepts of the theory of automata will be recalled or introduced. These notions are referred to in § 2 and § 12 only.

The notion of the Moore automaton is well-known, we denote such an automaton by $\mathbf{A}=(A, X, Y, \delta, \lambda)$.

Let a, b be two states; the length of a shortest (input) word p such that $\lambda(\delta(a, p)) \neq \lambda(\delta(b, p))$ is denoted by $\omega(a, b)$. The maximum of $\omega(a, b)$ (taken for pairs of different states) is called the *complexity* of $\mathbf{A}$.

Let us define the partitions[7] η_k in its state set A in such a manner that $a \equiv b$

[7] It follows from Proposition 16 of [2] that each η_k is really a partition. In [2], I have written R_k instead of η_k.

(mod η_k) holds exactly when $\omega(a, b) \geqq k$. It is obvious that

$$\eta_0 \supseteqq \eta_1 \supseteqq \eta_2 \supseteqq \eta_3 \supseteqq \cdots \tag{5.1}$$

and η_0 equals the maximal partition of A.

If we consider an automaton **A** such that the output set Y and the output function λ are not taken into account, then we speak of an *automaton without output signs*.

Let **A** be an automaton. Let us construct a directed graph G in the following way:

the states of **A** are the vertices of G, and

$\alpha_G(a, b)$ holds if and only if there is at least one $x(\in X)$ satisfying $\delta(a, x) = b$.

Then G is called the *graph of the automaton* **A**. (It is clear that we have regarded **A** as an automaton without output signs in this definition.)

II. Partitions having the property P in functional graphs

§ 6.

Construction I. Let G be a functional graph, B a basic set in G and d a divisor of the length of the cycle of G.

We form an augmenting sequence

$$G_1, G_2, \ldots, G_t \tag{6.1}$$

of induced subgraphs of G such that

(α) the vertex set V_1 of G_1 equals the set of inner vertices (with respect to B),

(β) the vertex set V_i of G_i consists of the vertices a which satisfy $\varphi(a) \in V_{i-1}$ (V_{i-1} is the vertex set of G_{i-1}; $2 \leqq i \leqq t$),

(γ) the sequence (6.1) terminates when we reach G (in the form of G_t).[8]

Let us construct a sequence

$$\pi_1, \pi_2, \ldots, \pi_t = \pi \tag{6.2}$$

(of partitions) according to the following rules (A)—(F):

(A) π_i is a partition of V_i (where $1 \leqq i \leqq t$).

(B) *(Initial step)* Choose a cyclic vertex c of G. Let

$$a \equiv b \pmod{\pi_1}$$

hold for $a(\in V_1)$ and $b(\in V_1)$ exactly when

$$\chi(a, c) \equiv \chi(b, c) \pmod{d}.$$

(C) Suppose that the partition π_{i-1} (of V_{i-1}) has already been defined (where $2 \leqq i \leqq t$). Denote by τ_i the following partition of V_i: $a \equiv b \pmod{\tau_i}$ precisely if

either $a = b$,

or $a \in V_i - V_{i-1}, b \in V_i - V_{i-1}$ and $\varphi(a) \equiv \varphi(b) \pmod{\pi_{i-1}}$.

[8] It is clear that $V_2 = B \cup V_1$.

(D) Assume that the partition π_{i-1} (of V_{i-1}) has already been defined (where $2 \leqq i \leqq t$). Denote by π_i' the subsequent partition of V_i: $a \equiv b \pmod{\pi_i'}$ exactly if
either $a=b$,
or $a \in V_{i-1}, b \in V_{i-1}$ and $a \equiv b \pmod{\pi_{i-1}}$.

(E) *(Ordinary step)* Choose an arbitrary partition π_i^* of V_i such that $\pi_i^* \subseteq \tau_i$. Form the union $\pi_i^* \cup \pi_i'$ and denote it by π_i.

(F) The construction of the sequence (6.2) contains an initial step and $t-1$ ordinary steps.

Remarks. If $a \in V_{i-1}$ and $b \in V_i - V_{i-1}$, then $a \not\equiv b$ modulo any of the partitions $\pi_i, \pi_{i+1}, \ldots, \pi_t (=\pi)$. — If $a \in V_i$ and $b \in V_i$, then either the congruence $a \equiv b$ is true for all of the partitions $\pi_i, \pi_{i+1}, \ldots, \pi_t$ or it is false for all of them. — The following three assertions are equivalent (for a performance of Construction I):
(1) B is empty,
(2) every vertex is inner and the construction collapses to the initial step,
(3) G/π is a cycle.
If G is a cycle, then the assertions (1), (2), (3) are true. — If B is empty and $d=1$, then π equals the maximal partition ι of V. — If B is chosen as the set of all acyclic vertices a fulfilling the statement that $\varphi(a)$ is cyclic, the number d is chosen as the cycle length of G and each π_i^* is the minimal partition of V_i, then π equals the minimal partition o of V.

Lemma 4. *The initial step of Construction I is independent of the choice of the cyclic vertex c.*

Proof. Apply the initial step with $c^{(1)}$ and $c^{(2)}$, resp. (instead of c). Denote $\chi(c^{(1)}, c^{(2)})$ by q. Let the originating partitions be $\pi_1^{(1)}$ and $\pi_1^{(2)}$.
Suppose $a \equiv b \pmod{\pi_1^{(1)}}$. Denote

$$\frac{\chi(a, c^{(1)}) - \chi(b, c^{(1)})}{d}$$

by k and p/d by m where p is the length of the cycle of G. It is easy to see that $\chi(a, c^{(2)})$ equals either $\chi(a, c^{(1)})+q$ or $\chi(a, c^{(1)})-(p-q)$ and a similar assertion holds with b (instead of a). A discussion shows that $\chi(a, c^{(2)}) - \chi(b, c^{(2)})$ is equal to one of $kd, (k+m)d, (k-m)d$. Hence $\chi(a, c^{(2)}) \equiv \chi(b, c^{(2)}) \pmod{d}$ and $a \equiv b \pmod{\pi_1^{(2)}}$.
An analogous inference shows that $a \equiv b \pmod{\pi_1^{(2)}}$ implies $a \equiv b \pmod{\pi_1^{(1)}}$.

Proposition 2. *Consider two performances of Construction I for a graph G; suppose that we start with the pairs $(B^{(1)}, d^{(1)})$ and $(B^{(2)}, d^{(2)})$, respectively. Denote the obtained partitions by $\pi^{(1)}$ and $\pi^{(2)}$. If $\pi^{(1)} = \pi^{(2)}$, then $B^{(1)} = B^{(2)}$, $d^{(1)} = d^{(2)}$ and the two performances are stepwise coinciding.*

Proof. We verify the statement indirectly.
If $B^{(1)} \neq B^{(2)}$, then there is a vertex a which is inner with respect to one of $B^{(1)}, B^{(2)}$ (e.g. to $B^{(1)}$) and outer with respect to the other one. Thus $a \equiv b \pmod{\pi^{(1)}}$ is satisfiable with at least one cyclic vertex b, but $a \equiv b \pmod{\pi^{(2)}}$ is not satisfiable by any cyclic b.

Let $d^{(1)}, d^{(2)}$ be different, we can suppose $d^{(1)} < d^{(2)}$. Choose a cyclic vertex a. a and $\varphi^{d^{(1)}}(a)$ are congruent modulo $\pi^{(1)}$ but they are incongruent modulo $\pi^{(2)}$.

Finally, we consider the case when $B^{(1)} = B^{(2)}$, $d^{(1)} = d^{(2)}$ and the two performances of Construction I differ from each other. The first difference between them will appear in the following manner: in two ordinary steps (corresponding to each other in the performances), $\pi_i^{*(1)}, \pi_i^{*(2)}$ act differently on the set $V_i - V_{i-1}$. It is evident (by the second sentence of the remarks above) that the partitions $\pi^{(1)}, \pi^{(2)}$ act on $V_i - V_{i-1}$ in the same manner as $\pi_i^{*(1)}$ and $\pi_i^{*(2)}$, respectively.

We have got $\pi^{(1)} \neq \pi^{(2)}$ when the two performances do not agree with each other completely.

§ 7.

Theorem 1. *The following three assertions are equivalent for a partition π of the vertex set V of a functional graph G:*

(I) *G/π is a functional graph,*

(II) *π has the property* P,

(III) *π can be obtained by Construction* I.

Proof. The equivalence of (I) and (II) follows immediately from Proposition 1 and Lemma 2. In what follows, we strive to show the equivalence of (II) and (III).

(II)⇒(III). Let us start with a P-partition π of V. Our aim is to determine a performance of Construction I such that π is obtained by this performance. In details the determination of the performance will consist of the following phases (α)—(ε):

(α) we determine a basic set B,

(β) we determine a divisor d of the length of the cycle of G,

(γ) we prove that if we choose two vertices a_1, a_2 and two elements b_1, b_2 of B such that the numbers $\chi(a_1, b_1)$ and $\chi(a_2, b_2)$ are defined and they do not coincide, then $a_1 \not\equiv a_2 \pmod{\pi}$,

(δ) we determine the partitions $\pi_2^*, \pi_3^*, \pi_4^*, \ldots$,

(ε) we show that each π_i^* is a refinement of τ_i.

We turn to elaborate the parts of the proof (of (II)⇒(III)) exposed above.

(α) Denote by C the set of all vertices a of G such that $[a]_\pi$ contains at least one cyclic vertex. Denote by B the set of vertices b such that $b \notin C$ and $\varphi(b) \in C$ are valid. It is clear that B consists of acyclic vertices. We are going to show that to any $b(\in B)$ no positive i can satisfy $b \equiv \varphi^i(b) \pmod{\pi}$. Suppose the contrary. It is easy to see (by the property P) that $b, \varphi^i(b), \varphi^{2i}(b), \varphi^{3i}(b), \ldots$ belong to a common class modulo π, this is impossible since $[b]_\pi$ cannot contain a cyclic vertex.

(β) Let a be an element of C, denote by $\eta(a)$ the smallest positive integer i such that $a \equiv \varphi^i(a) \pmod{\pi}$.

Consider a vertex $a(\in C)$, let i be the (minimal) number occurring in the definition of $\eta(a)$. Then

$$\varphi(a) \equiv \varphi(\varphi^i(a)) = \varphi^i(\varphi(a)) \pmod{\pi},$$

hence

$$\eta(a) \geqq \eta(\varphi(a)). \tag{7.1}$$

If (7.1) is applied for the vertices of the cycle of the graph, we get easily that $\eta(a)$ is common for the *cyclic* vertices. Denote this common value by d and the cycle length by p.

Our next aim is to verify that d is a divisor of p. Let k be the smallest integer such that $kd > p$. Since the deduction

$$c \equiv \varphi^d(c) \equiv \varphi^{2d}(c) \equiv \varphi^{3d}(c) \equiv \ldots \equiv \varphi^{kd}(c) = \varphi^{kd-p}(c) \pmod{\pi}$$

holds for an arbitrary cyclic vertex c, we have $kd-p \geqq \eta(c)(=d)$ by the minimality condition in the definition of η. On the other hand, the minimality condition in the definition of k implies $kd-p \leqq d$. Consequently $kd-p=d$, thus $(k-1)d=p$ and $d|p$.

Consider now an acyclic vertex $a(\in C)$, let c be a cyclic vertex such that $a \equiv c \pmod{\pi}$. We have $\varphi^i(a) \equiv \varphi^i(c) \pmod{\pi}$ for every i, this fact implies $\eta(a) \equiv \eta(c) = d$.

(γ) We can suppose $\chi(a_1, b_1) < \chi(a_2, b_2)$ without an essential restriction of the generality. Denote $\chi(a_1, b_1)$ by j. It is clear that $\varphi^{j+1}(a_1) \in C$ and $\varphi^{j+1}(a_2) \notin C$, hence $a_1 \not\equiv a_2 \pmod{\pi}$.

(δ) Denote by V_i (where $i \geqq 1$) the set of vertices a satisfying $\varphi^{i-1}(a) \in C$. (It is clear that $V_i \supseteqq V_{i-1}$ if $i \geqq 2$.) Let π_i^* be a partition of V_i defined by what follows: $a \equiv b \pmod{\pi_i^*}$ (where $a \in V_i$, $b \in V_i$) if and only if

$$\begin{aligned} &\text{either} \quad a = b \\ &\text{or} \qquad a \notin V_{i-1},\ b \notin V_{i-1} \ \text{ and } \ a \equiv b \pmod{\pi}. \end{aligned}$$

(ε) is obviously true with the above definition of the partitions π_i^*.

We have completed the determination of the "parameters" B, d and $\pi_2^*, \pi_3^*, \ldots$ occurring in Construction I. A routine inference shows (together with (γ)) that we obtain just π if we perform the construction with these "parameters".

(III)$\Rightarrow$(II). Consider a partition π which has been obtained by Construction I. Similarly to the preceding part of the proof, we denote by C the set of those vertices a for which $[a]_\pi$ contains a cyclic vertex.

Suppose $a \equiv b \pmod{\pi}$ where $a \neq b$.

If $a \in C$, then clearly $b \in C$. Let us choose an arbitrary cyclic vertex c. Either

$$\chi(\varphi(a), c) = \chi(a, c) - 1$$

or

$$\chi(\varphi(a), c) = p - 1 \equiv -1 = \chi(a, c) - 1 \pmod{d}$$

(according as $a \neq c$ or $a = c$), and the analogous statement holds for b (instead of a). Therefore we have

$$\chi(\varphi(a), c) \equiv \chi(a, c) - 1 \equiv \chi(b, c) - 1 \equiv \chi(\varphi(b), c) \pmod{d},$$

thus $\varphi(a) \equiv \varphi(b) \pmod{\pi}$.

If a and b do not belong to C, then they are necessarily contained in the same difference set $V_i - V_{i-1}$. $a \equiv b$ is valid modulo each of $\pi = \pi_t, \pi_{t-1}, \pi_{t-2}, \ldots, \ldots, \pi_i, \pi_i^*$ and τ_i (by the construction). We get $\varphi(a) \equiv \varphi(b) \pmod{\pi_{i-1}}$ by the rule (C), hence the elements $\varphi(a)$ and $\varphi(b)$ of V_{i-1} are congruent modulo each of $\pi_i, \pi_{i+1}, \ldots, \pi_t = \pi$, too.

The fulfilment of the property P is proved.

The next assertion is an easy consequence of the procedure described in Construction I and of the notion of factor graph:

Proposition 3. *Let G be a functional graph and π be a partition (in G) produced by Construction I. The cycle length of the factor graph G/π equals d. G/π is a cycle if and only if B is empty.*

III. Partitions having the property P in arbitrary sink-free graphs

§ 8.

Let G be a directed graph. We introduce a quaternary relation $\varkappa$ and some binary relations in the set V of vertices of G.

Let $\varkappa(a, b, c, d)$ hold for the (not necessarily different) vertices a, b, c, d if there is a positive integer k and there exist $2k$ vertices $f_1, f_2, \ldots, f_k, g_1, g_2, \ldots, g_k$ such that the equalities

$$a = f_1, \quad b = f_k, \quad c = g_1, \quad d = g_k \tag{8.1}$$

and the $2k-2$ relations

$$\alpha(f_1, f_2), \ \alpha(f_2, f_3), \ \ldots, \ \alpha(f_{k-1}, f_k), \tag{8.2}$$

$$\alpha(g_1, g_2), \ \alpha(g_2, g_3), \ \ldots, \ \alpha(g_{k-1}, g_k) \tag{8.3}$$

are true. $\varkappa(a, a, c, c)$ is regarded to be always valid (with the choice $k=1$) both when $a=c$ and when $a \neq c$. It is clear that $\varkappa(a, b, c, d)$ and $\varkappa(c, d, a, b)$ are equivalent.

Let $\varrho(a, b)$ be true if there is a $c (\in V)$ such that $\varkappa(c, a, c, b)$. Denote the transitive extension of ϱ by ε.

In Chapter III, our aim is to characterize the P-partitions of the sink-free graphs by use of the partition ε.

Remark. If G is a functional graph, then $\varepsilon = o$.

§ 9.

Lemma 5. *If π_1 and π_2 are partitions with property* P, *then $\pi_1 \cap \pi_2$ is a* P*-partition, too.*

Proof. If $a \equiv b \ (\mathrm{mod}\ \pi_1 \cap \pi_2)$, $\alpha(a, c)$ and $\alpha(b, d)$ are true, then both of $c \equiv d \ (\mathrm{mod}\ \pi_1)$, $c \equiv d \ (\mathrm{mod}\ \pi_2)$ hold.

Proposition 4. *There is a (uniquely determined)* P*-partition π^* such that $\pi^* \subseteqq \pi$ for each* P*-partition.*

Proof. G is a finite graph, hence the intersection π^* of all P-partitions possesses property P by a successive application of Lemma 5.

Proposition 5. *We have* $\pi^* \supseteq \varepsilon$.

Proof. Let a, b be two vertices such that $\varrho(a, b)$. There is a vertex c such that $\varkappa(c, a, c, b)$. Consider the $2k$ vertices occurring in (8.2), (8.3). (These vertices fulfil now $c=f_1=g_1$, $a=f_k$, $b=g_k$ instead of (8.1).) Since π^* has property P, $f_i \equiv g_i \pmod{\pi^*}$ follows inductively; especially,

$$a = f_k \equiv g_k = b \pmod{\pi^*}.$$

We have shown that $\varrho(a, b)$ implies $a \equiv b \pmod{\pi^*}$. Consequently, $a \equiv b \pmod{\varepsilon}$ implies $a \equiv b \pmod{\pi^*}$ (because ε is the transitive extension of ϱ).

Lemma 6. *If* G *has no sink, then* ε *is a* P*-partition.*

Proof. Assume $\varrho(a, b)$, $\alpha(a, c)$ and $\alpha(b, d)$ for some vertices a, b, c, d. Then there is a vertex h such that $\varkappa(h, a, h, b)$, hence $\varkappa(h, c, h, d)$, thus $\varrho(c, d)$.

Suppose $a \equiv b \pmod{\varepsilon}$, $\alpha(a, c)$, $\alpha(b, d)$ for an arbitrary quadruple a, b, c, d. There exist vertices $a_1, a_2, \ldots, a_k$ such that $\varrho(a_{i-1}, a_i)$ for each i $(2 \leqq i \leqq k)$ and $a_1 = a$, $a_k = b$. We can choose $k-2$ vertices $c_2, c_3, \ldots, c_{k-1}$ such that $\alpha(a_i, c_i)$ holds $(2 \leqq i \leqq k-1)$. By the beginning sentences of the proof, $\varrho(c_{i-1}, c_i)$ if $3 \leqq i \leqq k-1$; furthermore, $\varrho(c, c_2)$ and $\varrho(c_{k-1}, d)$. Therefore $\varepsilon(c, d)$.

Proposition 6. *If the directed graph* G *has no sink, then* $\pi^* = \varepsilon$.

Proof. $\pi^* \supseteq \varepsilon$ was stated in Proposition 5. $\pi^* \subseteq \varepsilon$ is an immediate consequence of Proposition 4 and Lemma 6.

Propositions 1, 6 and Lemma 2 imply

Corollary 1. *If* G *has no sink, then* G/ε *is a functional graph.*

Construction II. Let G be a graph without sinks. Denote the factor graph G/ε by G^*. Choose a partition π' of the vertex set of G^* such that π' is obtained by Construction I. Define a partition π in the vertex set V of G in the following manner: $a \equiv b \pmod{\pi}$ holds for $a (\in V)$, $b (\in V)$ exactly when $[a]_\varepsilon \equiv [b]_\varepsilon \pmod{\pi'}$.

Theorem 2. *Let* G *be a directed graph without sinks. The following three assertions are equivalent for a partition* π *of the vertex set of* G:

(i) G/π *is a functional graph,*

(ii) π *has the property* P,

(iii) π *can be obtained by Construction II.*

Proof. The theorem becomes clear by comparing the following earlier results: Theorem 1, Propositions 1, 4, 6, Corollary 1, Lemmas 1 and 2. (Now Lemma 1 is applied for ε and π instead of π_1 and π_2, resp.)

Proposition 7. *Let* G *be a graph without sinks. The length* p *of the cycle of the functional graph* G/ε *divides the greatest common divisor* p^* *of all cycle lengths of* G.

Sketch of the *proof.* Choose an arbitrary cycle Z' in G, denote the length of Z' by p'. Let us start with a vertex of Z' and pass through all the vertices of Z'; consider the corresponding vertices of G/ε. We have passed through the cycle Z of G/ε either one or more times; in any case, the number of surroundings of Z is an *integer*. Thus $p|p'$.

Since the same assertion holds for each choice of Z', we have $p|p^*$.

Corollary 2. *Let G be a graph without sinks. Then the following two numbers are equal:*

(a) *the number of partitions π such that G/π is a cycle,*

(b) *the number of divisors of the cycle length p of G/ε (including* 1 *and p).*

Proof. Recall Construction II, Theorem 2 and Proposition 3. It is clear that G/π is a cycle if and only if π' is constructed (in G/ε) by such a performance of Construction I that B is empty. This means that we have (precisely) the freedom of choosing a divisor of the cycle length of G/ε arbitrarily.

IV. Remarks, examples; an appendix

§ 10.

1. In the previous sections, a complete description of the partitions having property P of connected finite directed graphs without sinks was obtained. In the present remark, we shall outline how this description can be extended to non-connected graphs.

Let G be a non-connected directed graph containing no sink. Then G can be represented (in at least one manner) as the disjoint union of two graphs[9] G_1, G_2. Consider a partition π of the vertex set V of G; denote by π_i (where i can be 1 or 2) the restriction of π to the vertex set V_i of G_i. Let $[a]_\pi$ be an arbitrary π-class; evidently, either $[a]_\pi=[a]_{\pi_1}$ or $[a]_\pi=[a]_{\pi_2}$ (where necessarily $a\in V_1$ or $a\in V_2$, resp.) or $[a]_\pi=[b_1]_{\pi_1}\cup[b_2]_{\pi_2}$ with suitable vertices $b_1(\in V_1)$ and $b_2(\in V_2)$. It is easy to see the validity of the following assertion:

Proposition 8. *A partition π of V has property* P *if and only if*

π_1, π_2 are P*-partitions, and*

whenever $\alpha(a, b)$ holds in G and $[a]_\pi$ is the union of a π_1-class and a π_2-class, then the same statement holds for $[b]_\pi$, too.

The above idea can be utilized in such a way that first we form G/ε (which is clearly the disjoint union of G_1/ε and G_2/ε), we apply the proposition for $G/\varepsilon, G_1/\varepsilon$ and G_2/ε (instead of G, G_1, G_2, resp.), and we form the P-partitions of G by using the P-partitions of G/ε (analogously to Construction II).

2. The exposed theory admits a dualization with respect to reversing the orientation of edges. (The dual of a functional graph is a graph in which all in-degrees are one. Sources and sinks are dual to each other. The duals of the P-partitions are the partitions satisfying

$$(c \equiv d \,(\mathrm{mod}\,\pi) \;\&\; \alpha(a, c) \;\&\; \alpha(b, d)) \Rightarrow a \equiv b \,(\mathrm{mod}\,\pi).$$

[9] Each connected component of G is either a connected component of G_1 or a connected component of G_2.

The dual of $\varrho(a, b)$ is true exactly if there is a c such that $\varkappa(a, c, b, c)$ holds. And so on.)

3. It can be shown that the P-partitions of a sink-free graph form a lattice. The maximal element of this lattice is ι, its minimal element is ε.

4. In [7], [9], [10] also "input-independent partitions" have been studied. This notion is a slight modification of the concept of "partition with substitution property" (i.e., with property P). In our terminology, a partition π is called input-independent when

$$\big(\alpha(a, c) \ \& \ \alpha(a, d)\big) \Rightarrow c \equiv d \,(\mathrm{mod}\, \pi)$$

is universally true.

It is easy to see that this property is satisfied exactly when $\pi \supseteqq \varepsilon^*$ holds where ε^* is the transitive extension of the following relation ϱ^*: $\varrho^*(a, b)$ is valid if either $a=b$ or there exists a c such that $\alpha(c, a)$ & $\alpha(c, b)$.

5. We finish the section with exposing two open questions.

Problem 1. Let an overview of the P-partitions of the finite directed graphs containing sinks be given.

Problem 2. When does $p=p^*$ hold in Proposition 7?

§ 11.

In this section, we shall see some examples. The first example is used for illustrating how Constructions I, II are performed. This example and the two subsequent ones will serve for deciding the following questions:

(A) Is the relation ϱ always transitive, or is it really needed that it should be extended transitively? (Cf. § 8.)

(B) Can it happen that $\pi \supset \varepsilon$ for a sink-free graph, but π does not possess property P? (Cf. Propositions 4, 6.)

(C) Is the condition that sinks are not allowed indispensable in Proposition 6?

(D) Is $p<p^*$ possible in Proposition 7?

First, let us consider the graph G_1 seen on Fig. 1a. Since $\varrho(c, f)$ and $\varrho(f, g)$ are valid but $\varrho(c, g)$ does not hold, the transitive extension is a proper step when ε is formed. The classes modulo ε are:

$$\{a\}, \ \{b, d\}, \ \{c, f, g\}, \ \{e\}.$$

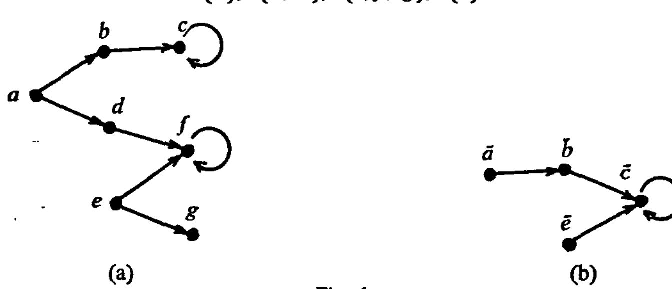


Fig. 1.

Fig. 1b shows the factor graph G_1/ε. (We write e.g. $\bar{a}$ instead of $[a]_\varepsilon$.) In G_1/ε, there is only one choice for d, namely $d=1$. We have six possibilities for choosing B, and Construction I can be performed in eight manners, the resulting partitions are seen on Table 1. (If $|B|=2$, then we have two possibilities for the choice of π_2^*, because τ_2 is the maximal partition of B.) The vertex set of G_1/ε has fifteen partitions; the remaining seven ones — among these,

$$\langle\{\bar{a}, \bar{c}\}, \{\bar{b}\}, \{\bar{e}\}\rangle \tag{11.1}$$

— do not have property P.

Table 1.

The elements of B	The classes modulo π
—	$\{\bar{a}, \bar{b}, \bar{c}, \bar{e}\}$
$\bar{a}$	$\{\bar{a}\}, \{\bar{b}, \bar{c}, \bar{e}\}$
$\bar{b}$	$\{\bar{a}\}, \{\bar{b}\}, \{\bar{c}, \bar{e}\}$
$\bar{e}$	$\{\bar{a}, \bar{b}, \bar{c}\}, \{\bar{e}\}$
$\bar{a}, \bar{e}$	$\{\bar{a}\}, \{\bar{e}\}, \{\bar{b}, \bar{c}\}$
	$\{\bar{a}, \bar{e}\}, \{\bar{b}, \bar{c}\}$
$\bar{b}, \bar{e}$	$\{\bar{a}\}, \{\bar{b}\}, \{\bar{e}\}, \{\bar{c}\}$
	$\{\bar{a}\}, \{\bar{b}, \bar{e}\}, \{\bar{c}\}$

Let us apply Construction II (with the partitions π of G_1/ε in the role of π'), we get that G_1 has eight P-partitions (from among the 15 partitions π fulfilling $\pi\supseteqq\varepsilon$). E.g.,

$$\langle\{a\}, \{b, c, d, f, g\}, \{e\}\rangle$$

is a P-partition of G_1 (obtained from the fifth row of Table 1), but

$$\langle\{a, c, f, g\}, \{b, d\}, \{e\}\rangle \tag{11.2}$$

(got from (11.1)) is not a P-partition; in fact, $\alpha(a, b), \alpha(c, c)$ hold and $a\equiv c$ but $b\not\equiv c$ modulo the partition (11.2).

The relation ε for the graph G_2 in Fig. 2 (containing three sinks) has the following equivalence classes:

$$\{a\}, \{b\}, \{c, d, e\}, \{f\}, \{g\}.$$

ε does not possess property P because $c\equiv e$ but $f\not\equiv g \pmod{\varepsilon}$. Therefore $\varepsilon\neq\pi^*$ in G_2.

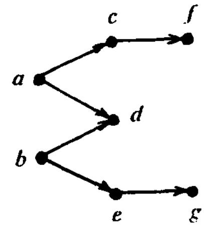
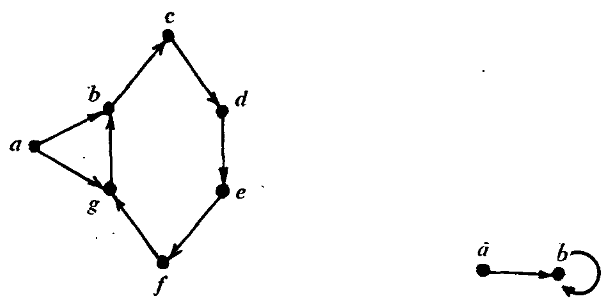


Fig. 2.

Consider the graph G_3 in Fig. 3a. ε has (on G_3) the equivalence classes

$$\{a\},\ \{b, c, d, e, f, g\}.$$

G_3/ε is seen in Fig. 3b. We can observe that $p=1<6=p^*$ (with the notations of Proposition 7).



(a) (b)

Fig. 3.

Summarizing, the examples show that the answers to the questions (A), (B), (C), (D) are: "the transitive extension is really needed", "yes", "yes", "yes", respectively.

Each counter-example given above contains a source.

Problem 3. Do the above answers to the questions (A)—(D) remain unchanged when we restrict ourselves to graphs without sources?

§ 12. (Appendix)

In this section our aim is to give a simple proof[10] for Conjecture 1 posed in [4].

Lemma 7. *Consider the sequence* $\eta_0, \eta_1, \eta_2, \ldots$ *of partitions of the state set* A *of a finite Moore automaton* $\mathbf{A}=(A, X, Y, \delta, \lambda)$. *If* $\eta_{i-1}=\eta_i$ *for some positive* i, *then* $\eta_i=\eta_{i+1}$.

[10] It should be noted that the idea of the present considerations is similar to a thought occurring in [8], p. 14.

Proof. Suppose $\eta_i \supset \eta_{i+1}$, we are going to show $\eta_{i-1} \supset \eta_i$. The supposition means that there are two states a, b such that $\omega(a, b)=i$. We have

$$\omega(\delta(a, x),\ \delta(b, x)) \geqq i-1 \tag{12.1}$$

for each choice of $x(\in X)$ and there is an $x^*(\in X)$ for which equality holds in (12.1). The state pair $\delta(a, x^*), \delta(b, x^*)$ is congruent modulo η_{i-1} but incongruent modulo η_i.

The next assertion is an easy consequence of Lemma 7.

Lemma 8. *Let* **A** *be as in the preceding lemma, denote* $|A|$ *by* v. *Let* m *be the smallest number such that* $\eta_m = \eta_{m+1}$. *Then* $m \leqq v-1$.

Lemma 9. *Let* **A**, v, m *be as in Lemmas* 7, 8. *If two states* a, b *satisfy* $\omega(a, b) \geqq v-1$, *then* $\omega(a, b)=\infty$.

Proof. Assume that a, b are congruent modulo η_{v-1}. They are congruent modulo η_m by Lemma 8 and (5.1); consequently, by the definition of m and Lemma 7, they are congruent modulo each of $\eta_{m+1}, \eta_{m+2}, \eta_{m+3}, \ldots$ (ad infinitum).

Proposition 9 ([4], Conjecture 1). *Let* **A** *be a finite Moore automaton such that the number* v *of its states satisfies* $v \geqq 2$. *Denote the complexity of* **A** *by* k. *If* k *is finite, then* $k \leqq v-2$.

Proof. By the finiteness of k, $\omega(a, b)$ is infinite (if and) only if $a=b$. Lemma 9 assures $\omega(a, b) \leqq v-2$ whenever $a \neq b$.

Corollary 3 of [3] shows that Proposition 9 cannot be sharpened.

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