On ranges of compositions of deterministic root-to-frontier tree transformations

Z. FÜLÖP and S. VÁGVÖLGYI

1. Introduction

In [3] we have proved that $\mathcal{DR}^2=\mathcal{DR}^n$ for every $n\geqq 2$ where $\mathcal{DR}$ is the class of all deterministic root-to-frontier tree transformations. This result motivated us for examining whether the set $S=\{\mathcal{DR}, \mathcal{NDR}, \mathcal{LDR}, \mathcal{LNDR}, \mathcal{H}, \mathcal{NH}, \mathcal{LH}\}$ generates, with composition $\circ$, a finite or infinite set of tree transformation classes. Here $\mathcal{H}$ is the class of all homomorphism tree transformations, moreover the linear, nondeleting and linear-nondeleting subclasses of a class are denoted by prefixing the class by $\mathcal{L}$, $\mathcal{N}$ and $\mathcal{LN}$, respectively. We note that the enlargement of S by $\mathcal{LNH}$ has no effect on the generated set $[S]=\{\mathcal{K}_1\circ\ldots\circ\mathcal{K}_n|n\geqq 1,\ \mathcal{K}_i\in S\ \text{ for }\ 1\leqq i\leqq n\}$ since, for each $\mathcal{C}\in S$, $\mathcal{C}\circ\mathcal{LNH}=\mathcal{LNH}\circ\mathcal{C}=\mathcal{C}$.

In Theorem 12 of [3] we obtained a characterization for the set $[S]$, by means of which we proved that $[S]$ is infinite if and only if the hierarchy $\{(\mathcal{LNDR}\circ\mathcal{NH})^n\}$ is proper, which was shown in [6].

In this paper we examine the set of surface set classes $[S](\mathcal{R}ec)=\{\mathcal{C}(\mathcal{R}ec)|\mathcal{C}\in[S]\}$ as well as the set of classes of tree transformation languages $\mathrm{yd}([S](\mathcal{R}ec))=\{\mathrm{yd}(\mathcal{T})|\mathcal{T}\in[S](\mathcal{R}ec)\}$. ($\mathcal{R}ec$ is the class of all recognizable forests and yd is the operation "taking the string formed by the leaves" for trees.) We show that, although $\langle[S]\rangle, \subseteqq\rangle$, as a poset, contains unrelated classes, $[S](\mathcal{R}ec)$ forms a chain with respect to inclusion with least element $\mathcal{R}ec$ and greatest element $\mathcal{DR}(\mathcal{R}ec)$. We also prove that, in this chain, $\mathcal{NDR}(\mathcal{R}ec)$ is properly contained in $\mathcal{DR}(\mathcal{R}ec)$ while the problem whether $[S](\mathcal{R}ec)$ is finite or infinite remains open. However, we show that the chain $\langle\mathrm{yd}([S](\mathcal{R}ec)), \subseteqq\rangle$ consists of exactly three elements.

2. Preliminaries

This paper is sequel to [3] and [6]. For notions and notations the reader is advised to consult with these works. Here we recall only the main results of [3] and [6] and introduce the terminology used exclusively in this paper.

We specify a special function symbol ε of arity 0 which either belongs to a ranked alphabet F or not.

If $p \in T_F$ is a tree then the yield $\mathrm{yd}(p) \subseteq F_0^*$ of p is defined inductively as follows:

(a) for $p \in F_0$, $\mathrm{yd}(p)=\lambda$ if $p=\varepsilon$ and $\mathrm{yd}(p)=p$ otherwise;

(b) for $p=f(p_1, \ldots, p_m)$, with $f \in F_m$ and $p_1, \ldots, p_m \in T_F$, $\mathrm{yd}(p)=\mathrm{yd}(p_1)\ldots\mathrm{yd}(p_m)$.

We call the attention of the reader not to confuse $\mathrm{yd}(p)$ with $\mathrm{fr}(p)$ defined in [3] and [6] and called the frontier of a tree p.

Subsets of T_F are called forests. If $T \subseteq T_F$ is a forest then $\mathrm{yd}(T)=\{\mathrm{yd}(p) \mid p \in T\}$ and, for a class $\mathscr{T}$ of forests we put $\mathrm{yd}(\mathscr{T})=\{\mathrm{yd}(T) \mid T \in \mathscr{T}\}$.

In [6] we defined the set of paths $\mathrm{path}(p) \subseteq N^*$ for a tree $p \in T_F(Y)$. Here we shall consider two distinguished elements, the longest leftmost path $\mathrm{llp}(p)$ and the longest rightmost path $\mathrm{lrp}(p)$ of $\mathrm{path}(p)$ which are defined in the following way:

(a) if $p \in Y \cup F_0$ then $\mathrm{llp}(p)=\mathrm{lrp}(p)=\lambda$,

(b) if $p=f(p_1, \ldots, p_m)$ for some $m \geqq 1$, $f \in F_m$ and $p_1, \ldots, p_m \in T_F(Y)$ then $\mathrm{llp}(p)=1\ \mathrm{llp}(p_1)$ and $\mathrm{lrp}(p)=n\ \mathrm{lrp}(p_n)$.

Let $\tau \subseteq T_F \times T_G$ be a tree transformation. The range of τ, defined as usual, is denoted by $\mathrm{ran}(\tau)$. Let $T \subseteq T_F$ be a forest. The image $\tau(T)$ of T under τ is the set $\{q \in T_G \mid (p, q) \in \tau$ for some $p \in T\}$.

For a class $\mathscr{C}$ of tree transformations and a class $\mathscr{T}$ of forests we set $\mathrm{ran}(\mathscr{C})=\{\mathrm{ran}(\tau) \mid \tau \in \mathscr{C}\}$ and $\mathscr{C}(\mathscr{T})=\{\tau(T) \mid \tau \in \mathscr{C}$ and $T \in \mathscr{T}\}$.

We denote by $\mathscr{R}ec$ the class of all recognizable forests (c.f. [4]).

Again, let $\mathscr{C}$ be a class of tree transformations.

The class of surface sets of $\mathscr{C}$ is the class $\mathscr{C}(\mathscr{R}ec)$ of forests, moreover, the class of tree transformation languages of $\mathscr{C}$ is the class $\mathrm{yd}(\mathscr{C}(\mathscr{R}ec))$ of languages.

If $\tau \subseteq T_F \times T_G$ is a tree transformation then the tree-to-string transformation τ_{tts} underlying τ is $\tau_{\mathrm{tts}}=\{(p, \mathrm{yd}(q)) \mid (p, q) \in \tau\}$. Thus $\tau_{\mathrm{tts}} \subseteq T_F \times G_0^*$. Analogously, for a class $\mathscr{C}$ of tree transformations we define $\mathscr{C}_{\mathrm{tts}}=\{\tau_{\mathrm{tts}} \mid \tau \in \mathscr{C}\}$.

We recall that the composition $\mathscr{C}_1 \circ \mathscr{C}_2$ of two tree transformation classes was defined in the order "first $\mathscr{C}_1$ and then $\mathscr{C}_2$" (c.f. [3], [6]). Thus we have $(\mathscr{C}_1 \circ \mathscr{C}_2)_{\mathrm{tts}}=\mathscr{C}_1 \circ \mathscr{C}_{2\mathrm{tts}}$ and, for any class $\mathscr{T}$ of forests $\mathrm{yd}(\mathscr{C}_1(\mathscr{T}))=\mathscr{C}_{1\mathrm{tts}}(\mathscr{T})$.

Let $\{\mathscr{C}_n \mid n=1, 2, \ldots\}$ be a set of classes. We say that $\{\mathscr{C}_n \mid n=1, 2, \ldots\}$, or $\{\mathscr{C}_n\}$ for short, is a hierarchy if $\mathscr{C}_n \subseteq \mathscr{C}_{n+1}$ for each $n \geqq 1$. This hierarchy is proper if $\mathscr{C}_n \subset \mathscr{C}_{n+1}$.

Now we introduce some technical details which, hopefully, make easier to understand the proofs in this paper.

Consider a DR transducer $\mathfrak{A}=(F, A, G, P, a_0)$ and a rule $af(x_1, \ldots, x_m) \to q$ in P. In this paper q is considered as an element of $T_G(A \times X_m)$ rather than $T_G(A(X_m))$. This is important when speaking about the height $h(q)$ of the right-hand side of a rule. (For the definition of height, see [3] or [6].) Moreover, we extend yd for the elements $T_G(A \times X_m)$ as follows: $\mathrm{yd}(q)=q$ if $q \in A \times X_m$ and otherwise $\mathrm{yd}(q)$ is defined in the same way as if q were in T_G, see above. Thus if q is the right-hand side of the above rule then $\mathrm{yd}(q)$ can be written in the form $w_0(a_1, x_{i_1})w_1 \ldots (a_n, x_{i_n})w_n$ for some $n \geqq 0$, $w_0, \ldots, w_n \in G_0^*$, $a_1, \ldots, a_n \in A$ and $x_{i_1}, \ldots, x_{i_n} \in X_m$.

The length of a string w will be denoted by $|w|$. The following abbreviated notation will also be used. Let F and G be disjoint ranked alphabets, let $f \in F_m$ with $m \geqq 0$ and $w \in G_0^*$ with $w=a_1 \ldots a_m$ for some $a_1, \ldots, a_m \in G_0$. For any partition $w=w_1 \ldots w_n$ $(n \geqq 0)$ of w the notation $f(w_1, \ldots, w_n)$ stands for the tree $f(a_1, \ldots, a_m) \in T_{F \cup G}$.

Finally we restate the main results of [3] and [6].

Denote the set $\{\mathcal{DR}, \mathcal{NDR}, \mathcal{LDR}, \mathcal{LNDR}, \mathcal{H}, \mathcal{NH}, \mathcal{LH}\}$ of tree transformation classes by S. The set of all tree transformation classes generated by S with composition $\circ$ is $[S]=\{\mathcal{K}_1\circ\ldots\circ\mathcal{K}_n | n\geqq 1,\ \mathcal{K}_i\in S \text{ for } 1\leqq i\leqq n\}$.

Let us introduce, for each integer $k\geqq 0$, the class $\mathcal{C}_k$ of tree transformations as follows:

(a) $\mathcal{C}_0=\mathcal{LNDR}$,

(b) $\mathcal{C}_{k+1}=\mathcal{C}_k\circ\mathcal{NH}$ if k is even and $\mathcal{C}_{k+1}=\mathcal{C}_k\circ\mathcal{LNDR}$ if k is odd.

Moreover, consider the two finite subsets S_1 and S_2 of $[S]$ defined by

$$S_1 = S\cup\{\mathcal{DR}^2, \mathcal{LDR}\circ\mathcal{NH}, \mathcal{LDR}^2, \mathcal{LDR}\circ\mathcal{NDR}, \mathcal{H}\circ\mathcal{NDR}, \mathcal{LDR}^2\circ\mathcal{NDR}, \mathcal{LNDR}\circ\mathcal{H}\}$$

and

$$S_2 = \{\mathcal{H}, \mathcal{NH}, \mathcal{LH}, \mathcal{LDR}\circ\mathcal{NH}, \mathcal{LNDR}\circ\mathcal{H}\}.$$

Proposition 2.1. (Theorem 12 of [3].) For each $\mathcal{C}\in[S]$ one of the following three assertions holds:

(i) $\mathcal{C}\in S_1$,

(ii) $\mathcal{C}=\mathcal{C}_k$ for some $k\geqq 0$,

(iii) $\mathcal{C}=\mathcal{C}'\circ\mathcal{C}_k$ for some $\mathcal{C}'\in S_2$ and $k\geqq 0$.

By this proposition, $[S]$ is infinite if and only if the hierarchy $\{\mathcal{C}_k\}$ is proper. Then, in [6] we obtained the following result.

Proposition 2.2. (Theorem 3 of [6].) $\{\mathcal{C}_{2k+1} | k=0, 1, \ldots\}$ is a proper hierarchy.

Notice that it follows from Proposition 2.2 that $\{\mathcal{C}_k\}$ is also a proper hierarchy. This can easily be seen by using the identities $\mathcal{LNDR}\circ\mathcal{LNDR}=\mathcal{LNDR}$ and $\mathcal{NH}\circ\mathcal{NH}=\mathcal{NH}$.

3. The results

First we examine the set of surface set classes $[S](\mathcal{Rec})=\{\mathcal{C}(\mathcal{Rec}) | \mathcal{C}\in[S]\}$. We have the following result.

Theorem 3.1. The poset $\langle [S](\mathcal{Rec}), \subseteqq\rangle$ is a chain which can be written in the following form:

$\mathcal{Rec}\subseteqq\mathcal{NH}(\mathcal{Rec})\subseteqq\mathcal{NH}\circ\mathcal{C}_0(\mathcal{Rec})\subseteqq\mathcal{NH}\circ\mathcal{C}_1(\mathcal{Rec})\ldots\subseteqq\mathcal{NDR}(\mathcal{Rec})\subseteqq\mathcal{DR}(\mathcal{Rec})$.

Proof. By Proposition 2.1, we have $[S](\mathcal{Rec})=\{\mathcal{C}(\mathcal{Rec}) | \mathcal{C}\in S_1\}\cup\{\mathcal{C}_k(\mathcal{Rec}) | k\geqq 0\}\cup\{\mathcal{C}'\circ\mathcal{C}_k(\mathcal{Rec}) | \mathcal{C}'\in S_2 \text{ and } k\geqq 0\}$. Then, using the results $\mathcal{DR}^2(\mathcal{Rec})=\mathcal{DR}(\mathcal{Rec})$ (Theorem I. 3. in [5]) and $\mathcal{LDR}(\mathcal{Rec})=\mathcal{LNDR}(\mathcal{Rec})=\mathcal{LH}(\mathcal{Rec})=\mathcal{Rec}$ (Corollary IV.6.6. in [4]) as well as the identities $\mathcal{LH}\circ\mathcal{NH}=\mathcal{H}$ and $\mathcal{NH}\circ\mathcal{NDR}=\mathcal{NDR}$ ([3]) we can write

$$\{\mathcal{C}(\mathcal{Rec}) | \mathcal{C}\in S_1\} = \{\mathcal{Rec}, \mathcal{NH}(\mathcal{Rec}), \mathcal{NDR}(\mathcal{Rec}), \mathcal{DR}(\mathcal{Rec})\},$$

$$\{\mathcal{C}_k(\mathcal{Rec}) | k \geqq 0\} = \{\mathcal{Rec}, \mathcal{NH}(\mathcal{Rec}), \mathcal{NH}\circ\mathcal{C}_0(\mathcal{Rec}), \mathcal{NH}\circ\mathcal{C}_1(\mathcal{Rec}), \ldots\}$$

and

$$\{\mathcal{C}'\circ\mathcal{C}_k(\mathcal{Rec}) | \mathcal{C}'\in S_2 \text{ and } k \geqq 0\} = \{\mathcal{NH}(\mathcal{Rec}), \mathcal{NH}\circ\mathcal{C}_0(\mathcal{Rec}), \mathcal{NH}\circ\mathcal{C}_1(\mathcal{Rec}), \ldots\}$$

obtaining all the elements of $[S](\mathcal{Rec})$. For proving the inclusions stated in our theorem we only have to observe that, since $\mathcal{NDR}$ is closed under composition,

$\mathscr{C}_k \subseteq \mathscr{N}\mathscr{D}\mathscr{R}$ and thus $\mathscr{N}\mathscr{H} \circ \mathscr{C}_k \subseteq \mathscr{N}\mathscr{D}\mathscr{R}$ for each $k \geqq 0$. All the other inclusions follow by definition. □

We can raise the question that which of the inclusion relations appearing in Theorem 3.1 are proper. It is a folkloric result that $\mathscr{R}ec \subset \mathscr{N}\mathscr{H}(\mathscr{R}ec)$, moreover, it is also not difficult to see that $\mathscr{N}\mathscr{H}(\mathscr{R}ec) \subset \mathscr{N}\mathscr{H} \circ \mathscr{C}_0(\mathscr{R}ec)$ which, in our paper, will be a consequence of Theorem 3.6. The questions that whether the hierarchy $\{\mathscr{N}\mathscr{H} \circ \mathscr{C}_k(\mathscr{R}ec)\}$ of classes of surface sets is proper or not and that whether $\bigcup_{k=0}^{\infty} \mathscr{N}\mathscr{H} \circ \mathscr{C}_k(\mathscr{R}ec) \subset \mathscr{N}\mathscr{D}\mathscr{R}(\mathscr{R}ec)$ are much more interesting and, at the same time, difficult. These problems are still open. However, we obtained the following result:

Lemma 3.2. $\mathscr{N}\mathscr{D}\mathscr{R}(\mathscr{R}ec) \subset \mathscr{D}\mathscr{R}(\mathscr{R}ec)$.

Proof. We observe that, by Theorem 3.2.1 of [2], $\operatorname{ran}(\mathscr{D}\mathscr{R}) = \mathscr{D}\mathscr{R}(\mathscr{R}ec)$ and $\operatorname{ran}(\mathscr{N}\mathscr{D}\mathscr{R}) = \mathscr{N}\mathscr{D}\mathscr{R}(\mathscr{R}ec)$. Therefore it is sufficient to give a forest in $\operatorname{ran}(\mathscr{D}\mathscr{R})$ which is not in $\operatorname{ran}(\mathscr{N}\mathscr{D}\mathscr{R})$.

Let us introduce the ranked alphabet $F = F_0 \cup F_1 \cup F_2$ where $F_0 = \{\#\}$, $F_1 = \{f_1, f_2\}$ and $F_2 = \{g\}$. Denote the balanced tree of type $\{g, \#\}$ with height n by t'_n. Then construct the tree t_n from t'_n in the following manner: for each $w \in \operatorname{path}(t'_n)$ with $|w| = n$ substitute the tree $f_{i_1}(\dots f_{i_n}(\#)\dots)$ for $\operatorname{str}(t'_n, w)$ in t'_n where $w = i_1 \dots i_n$. (We know that, for such a w, $\operatorname{str}(t'_n, w) = \#$ and that $1 \leqq i_1, \dots, i_n \leqq 2$.) An example for the case $n = 2$ of this construction can be seen in Fig. 1.
With this we achieved that each subtree of t_n with root g has exactly one occurrence in t_n.

Next we take a function symbol f with arity 2 and two function symbols e and h with arity 1. Let $G = F \cup \{e\}$ and $H = F \cup \{f, h\}$.

There exists a DR transducer $\mathfrak{A}$ such that $\tau_{\mathfrak{A}} = \{(e(p), f(p, h^n(\#))) \mid p \in T_F$ and $n = |\operatorname{llp}(p)|\}$, where $h^n(\#) = \#$ if $n = 0$ and $h^n(\#) = h(h^{n-1}(\#))$ if $n \geqq 1$. (Notice that $\tau_{\mathfrak{A}} \subseteq T_G \times T_H$, moreover that $|\operatorname{llp}(q)| = |\operatorname{lrp}(q)|$ holds whenever $q \in \operatorname{ran}(\tau_{\mathfrak{A}})$.)

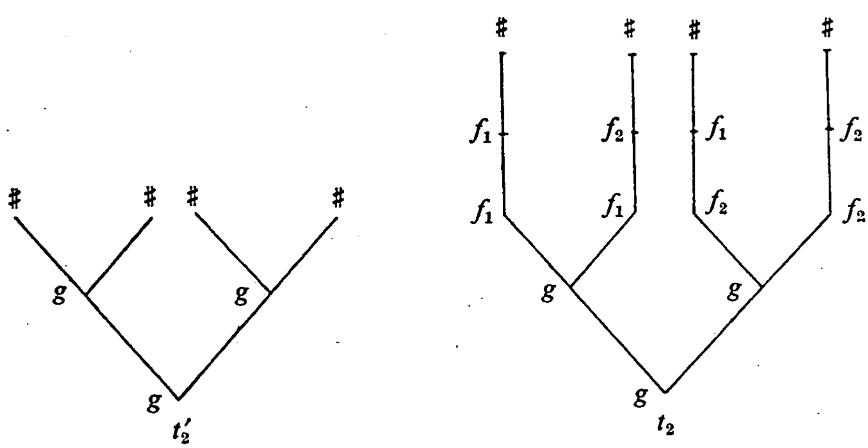

Figure 1.

In fact, the DR transducer the rules of which are listed below can be taken as $\mathfrak{A}$. The initial state is a.

$$ae(x_1) \to f(bx_1, cx_1),$$

$$bg(x_1, x_2) \to g(bx_1, bx_2),$$

$$bf_i(x_1) \to f_i(bx_1), \quad i = 1, 2, \quad b\# \to \#,$$

$$cg(x_1, x_2) \to h(cx_1), cf_i(x_1) \to h(cx_1), \quad i = 1, 2, \quad c\# \to \#.$$

We show that $\operatorname{ran}(\tau_{\mathfrak{A}}) \notin \operatorname{ran}(\mathcal{NDR})$. For this, let us introduce first the abbreviation $q_n = g(t_n, h^{2n}(\#))$, for $n \geq 1$. Then, since $\tau_{\mathfrak{A}}$ sends $e(t_n)$ to q_n we have that $\{q_n | n = 1, 2, \ldots\} \subseteq \operatorname{ran}(\tau_{\mathfrak{A}})$.

Now suppose indirectly that there exists an NDR transducer $\mathfrak{B} = (E, B, H, P, b_0)$ such that $\operatorname{ran}(\tau_{\mathfrak{A}}) = \operatorname{ran}(\tau_{\mathfrak{B}})$. Then also $\{q_n | n = 1, 2, \ldots\} \subseteq \operatorname{ran}(\tau_{\mathfrak{B}})$ therefore, for each $n = 1, 2, \ldots$ there exists a $p'_n \in T_E$ so that $b_0 p'_n \overset{*}{\underset{\mathfrak{B}}{\Rightarrow}} q_n$. We note that some of these derivations may start with such a sequence of rules in which the height of the right-hand side of each rule is 0. But, after dropping this sequence of rules from each derivation we have that for each $n = 1, 2, \ldots$ there exists a $b_n \in B$ and a $p_n \in \operatorname{sub}(p'_n)$ with $b_n p_n \overset{*}{\underset{\mathfrak{B}}{\Rightarrow}} q_n$ such that each derivation starts with a rule, the height of the right-hand side of which is greater than 0. Then we can choose an infinite subsequence $n_1, n_2, \ldots, n_k, \ldots$ of $1, 2, \ldots, n, \ldots$ such that the same rule, let us say $b\sigma(x_1, \ldots, x_u) \to q(b_1 x_{i_1}, \ldots, b_v x_{i_v})$ is applied in the first step of the derivations $b_{n_k} p_{n_k} \overset{*}{\underset{\mathfrak{B}}{\Rightarrow}} q_{n_k}$ for $k = 1, 2, \ldots$. (This, of course, entails that $b = b_{n_k}$ for each $k = 1, 2, \ldots$.) Moreover, without loss of generality, we may suppose that $q \in \hat{T}_{H,v}$ and $\operatorname{fr}(q) = x_1, \ldots x_v$. (For notations, see [3] or [6].)

We observe that the longest leftmost path (resp. longest rightmost path) of q ends in x_1 (resp. x_v) or, formally, $\operatorname{str}(\operatorname{llp}(q), q) = x_1$ (resp. $\operatorname{str}(\operatorname{lrp}(q), q) = x_v$). For, if this were not the case then $|\operatorname{llp}(q_{n_k})|$ (resp. $|\operatorname{lrp}(q_{n_k})|$) would be a constant for each $k = 1, 2, \ldots$.

Next we show that $x_{i_1} = x_{i_v}$ or, equivalently, $i_1 = i_v$. On the contrary, assume that $i_1 < i_v$. Choose two integers k and l such that $k < l$ and write the derivations $bp_{n_k} \overset{*}{\underset{\mathfrak{B}}{\Rightarrow}} q_{n_k}$ and $bp_{n_l} \overset{*}{\underset{\mathfrak{B}}{\Rightarrow}} q_{n_l}$ in more detailed form as

$$\begin{aligned} bp_{n_k} &= b\sigma(p_1^{(k)}, \ldots, p_{i_1}^{(k)}, \ldots, p_{i_v}^{(k)}, \ldots, p_u^{(k)}) \overset{*}{\underset{\mathfrak{B}}{\Rightarrow}} \\ &q(b_1 p_{i_1}^{(k)}, \ldots, b_v p_{i_v}^{(k)}) \overset{*}{\underset{\mathfrak{B}}{\Rightarrow}} q(q_1^{(k)}, \ldots, q_v^{(k)}) = q_{n_k} \end{aligned} \tag{1}$$

and similarly

$$\begin{aligned} bp_{n_l} &= b\sigma(p_1^{(l)}, \ldots, p_{i_1}^{(l)}, \ldots, p_{i_v}^{(l)}, \ldots, p_u^{(l)}) \overset{*}{\underset{\mathfrak{B}}{\Rightarrow}} \\ &q(b_1 p_{i_1}^{(l)}, \ldots, b_v p_{i_v}^{(l)}) \overset{*}{\underset{\mathfrak{B}}{\Rightarrow}} q(q_1^{(l)}, \ldots, q_v^{(l)}) = q_{n_l}. \end{aligned}$$

These two derivations entail that

$$b\sigma(p_1^{(k)}, \ldots, p_{i_1}^{(k)}, \ldots, p_{i_v}^{(l)}, \ldots, p_u^{(k)}) \overset{*}{\underset{\mathfrak{B}}{\Rightarrow}} q(q_1^{(k)}, \ldots, q_v^{(l)})$$

from where we see that $q(q_1^{(k)}, \ldots, q_v^{(l)}) \in \operatorname{ran}(\tau_{\mathfrak{B}})$ and thus, by $\operatorname{ran}(\tau_{\mathfrak{A}}) = \operatorname{ran}(\tau_{\mathfrak{B}})$,

$q(q_1^{(k)}, \ldots, q_v^{(l)}) \in \operatorname{ran}(\tau_{\mathfrak{A}})$. Then, by the note we made after the definition of $\tau_{\mathfrak{A}}$, $|\operatorname{llp}(q(q_1^{(k)}, \ldots, q_v^{(l)}))| = |\operatorname{lrp}(q(q_1^{(k)}, \ldots, q_v^{(l)}))|$. On the other hand

$$|\operatorname{llp}(q(q_1^{(k)}, \ldots, q_v^{(l)}))| = |\operatorname{llp}(q)| + |\operatorname{llp}(q_1^{(k)})| = |\operatorname{llp}(q_{n_k})| = 2n_k+1 \quad \text{and}$$

$$|\operatorname{lrp}(q(q_1^{(k)}, \ldots, q_v^{(l)}))| = |\operatorname{lrp}(q)| + |\operatorname{lrp}(q_v^{(l)})| = |\operatorname{lrp}(q_{n_l})| = 2n_l+1, \quad \text{that is,} \quad n_k = n_l.$$

This is a contradiction, since $k<l$.

Let us suppose that $i_1 = i_v = 1$.

Denote the number of states in B by $|B|$ and let $K = \max\{h(q) | q$ is the right-hand side of some rule in $P\}$. Let the integer k be chosen and fixed such that $n_k > K(|B|+1)$.

Consider, from (1), the derivation $b_v p_1^{(k)} \overset{*}{\underset{\mathfrak{B}}{\Rightarrow}} q_v^{(k)}$. Since $\operatorname{lrp}(q)$ ends in x_v, by the definition of q_{n_k}, $q_v^{(k)}$ contains only the function symbols h and $\#$ of H. But then, since $\mathfrak{B}$ is an NDR transducer and the arity of h is 1, the arity of the function symbols occuring in $p_1^{(k)}$ is either 1 or 0.

Consider now the derivation $b_1 p_1^{(k)} \overset{*}{\underset{\mathfrak{B}}{\Rightarrow}} q_1^{(k)}$. We state three properties of $q_1^{(k)}$. Namely, by the choice of k, we have

(P1) $h(q_1^{(k)}) \geq 2n_k + 1 - K > 2 \cdot |B| \cdot K$

moreover, by the position of $q_1^{(k)}$ in q_{n_k},

(P2) if $w \in \operatorname{path}(q_1^{(k)})$ is such that $\operatorname{lab}(q_1^{(k)}, w)$ is f_1, f_2 or $\#$ then $|w| > |B| \cdot K$

and, since $q_1^{(k)}$ is a subtree of t_{n_k},

(P3) each subtree of $q_1^{(k)}$ with root g has exactly one occurrence in $q_1^{(k)}$.

Further on, we analyse the derivation $b_1 p_1^{(k)} \overset{*}{\underset{\mathfrak{B}}{\Rightarrow}} q_1^{(k)}$. Therefore, consider the following algorithm.

let $i=0$, $r_0 = x_1$, $b_1^{(0)} = b_1$, $s_0 = p_1^{(k)}$, $m_0 = 1$;
while $r_i \neq q_1^{(k)}$ **do**
  begin
    search for the smallest integer j for which $r_i(b_1^{(i)} s_i, \ldots, b_{m_i}^{(i)} s_i) \overset{j}{\underset{\mathfrak{B}}{\Rightarrow}} r(b_1' s, \ldots, b_m' s)$ holds for some $m \geq 0$, $r \in \hat{T}_{H,m}$, $s \in T_E$ and $b_1', \ldots, b_m' \in B$ such that $\operatorname{rn}(r_i) < \operatorname{rn}(r)$;
    let $i = i+1$;
    let $r_i = r$, $s_i = s$, $m_i = m$, $j_i = j$
      and $b_l^{(i)} = b_l'$ for $1 \leq l \leq m$
  end

(Here $\overset{j}{\underset{\mathfrak{B}}{\Rightarrow}}$ stands for the j-fold composition of the relation $\underset{\mathfrak{B}}{\Rightarrow}$.)

We note that the smallest integer j in the above algorithm can be found by rewriting simultaneously the subtrees $b_1^{(i)} s_i, \ldots, b_{m_i}^{(i)} s_i$. (This simultaneous rewriting was called parallel derivation in [2].)

Since each derivation of $\mathfrak{B}$ starting from a state and an input tree terminates after a finite number of steps our algorithm also terminates after, let us say, N steps. Moreover, since $b_1 p_1^{(k)} \overset{*}{\underset{\mathfrak{B}}{\Rightarrow}} q_1^{(k)}$, it holds that $m_N = 0$ and $r_N = q_1^{(k)}$. Thus we can write

$$r_0(b_1^{(0)} s_0) \overset{j_1}{\underset{\mathfrak{B}}{\Rightarrow}} r_1(b_1^{(1)} s_1, \ldots, b_{m_1}^{(1)} s_1) \overset{j_2}{\underset{\mathfrak{B}}{\Rightarrow}} \ldots \overset{j_N}{\underset{\mathfrak{B}}{\Rightarrow}} r_N(b_1^{(N)} s_N, \ldots, b_{m_N}^{(N)} s_N) = q_1^{(k)}.$$

We make the following observations.

Since we choose the smallest integer j in the while loop it holds that $h(r_i) \leqq i \cdot K$, for $1 \leqq i \leqq N$, therefore, by property (P1) of $q_1^{(k)}$, we have that $N > 2 \cdot |B|$.

Let $i = |B|$. Then, by property (P2) of $q_1^{(k)}$ we obtain that each tree of $r_1, \ldots, r_i$ contains only the function symbol g of H. Thus the condition $\mathrm{rn}(r_0) < \mathrm{rn}(r_1) < \ldots < \mathrm{rn}(r_i)$ entails that $2 \leqq m_1 < \ldots < m_i$, hence, we get that $m_i > |B|$.

Then, for $i = |B|$, there is at least one state that appears at least twice in the sequence $b_1^{(i)}, \ldots, b_{m_i}^{(i)}$.

Since $r_i(b_1^{(i)} s_i, \ldots, b_{m_i}^{(i)} s_i) \overset{*}{\underset{\mathfrak{B}}{\Rightarrow}} q_1^{(k)}$ we obtain, by (P2) and $h(r_i) \leqq i \cdot K = B \cdot K$, that there is a subtree with root g of $q_1^{(k)}$ which appears at least twice in $q_1^{(k)}$. However, this contradicts property (P3) of $q_1^{(k)}$. With this we finished the proof of our lemma. □

We note that in the above proof we strongly used the fact that the output ranked alphabet H of our counter-example $\tau_{\mathfrak{A}}$ contains function symbols of arity 1. It is not clear how this lemma could be proved if we restricted ourselves to ranked alphabets that do not contain 1-ary function symbols.

Now we begin to deal with the poset $\langle \mathrm{yd}([S](\mathcal{Rec})), \subseteq \rangle$ where $\mathrm{yd}([S](\mathcal{Rec})) = \{\mathrm{yd}(\mathcal{T}) | \mathcal{T} \in [S](\mathcal{Rec})\}$. We observe that, since $\langle [S](\mathcal{Rec}), \subseteq \rangle$ is a chain and yd preserves inclusion, $\langle \mathrm{yd}([S](\mathcal{Rec})), \subseteq \rangle$ is also a chain. First we prove a technical lemma.

Lemma 3.3. $\mathcal{NDR}_{\mathrm{tts}} = \mathcal{DR}_{\mathrm{tts}}$.

Proof. It is sufficent to show that $\mathcal{DR}_{\mathrm{tts}} \subseteq \mathcal{NDR}_{\mathrm{tts}}$. To this end take a DR transducer $\mathfrak{A} = (F, A, G, P, a_0)$ and denote the number of rules in P by $|P|$. Suppose that the rules in P are numbered from 1 to $|P|$.

The following algorithm produces, for each $i = 1, \ldots, |P|$, a function symbol f_i and a rule ϱ_i for a DR transducer:

(a) Suppose that the i-th rule is of the form $af(x_1, \ldots, x_m) \to q$ where $q \in T_G(A \times X_m)$.

(b) Let $\mathrm{yd}(q) = w_0(a_1, x_{i_1}) w_1 \ldots (a_n, x_{i_n}) w_n$ where $n \geqq 0$, $1 \leqq x_{i_1}, \ldots, x_{i_n} \leqq m$, $w_0, w_1, \ldots, w_n \in G_0^*$.

(c) Let $\{x_{j_1}, \ldots, x_{j_k}\} \subseteq X_m$ be the set of all variables which do not occur in q (and so neither in $\mathrm{yd}(q)$).

(d) Let f_i be a new function symbol with arity $|w_0| + \ldots + |w_n| + n + k$.

(e) Let ϱ_i be the rule $af(x_1, \ldots, x_m) \to f_i(w_0, a_1 x_{i_1}, \ldots, a_n x_{i_n}, w_n, c x_{j_1}, \ldots, c x_{j_k})$ where $c \notin A$ is a new state. (As usual, (a_k, x_{i_k}) is abbreviated by $a_k x_{i_k}$, for $1 \leqq k \leqq n$.)

Now we introduce the DR transducer $\mathfrak{B} = (F, A \cup \{c\}, F', P', a_0)$ where

$$F' = \{f_i | i = 1, \ldots, |P|\} \cup F \cup \{\varepsilon\} \quad \text{and}$$

$$P' = \{\varrho_i | i = 1, \ldots, |P|\} \cup \{cf(x_1, \ldots, x_m) \to$$
$$\to f(cx_1, \ldots, cx_m) | m \geqq 1, \ f \in F_m\} \cup \{cf \to \varepsilon | f \in F_0\}.$$

It can be seen from the construction that $\mathfrak{B}$ is an NDR transducer. Moreover, it can be verified by an induction on p that for each $a \in A$, $p \in T_F$ and $w \in G_0^*$,

$$(\exists q \in T_G)(ap \overset{*}{\underset{\mathfrak{A}}{\Rightarrow}} q \wedge \mathrm{yd}(q) = w) \Leftrightarrow (\exists q' \in T_{F'})(ap \overset{*}{\underset{\mathfrak{B}}{\Rightarrow}} q' \wedge \mathrm{yd}(q') = w).$$

It then follows that $\tau_{\mathfrak{A}_{tts}}=\tau_{\mathfrak{B}_{tts}}$. Hence we have $\mathcal{DR}_{tts}\subseteq\mathcal{NDR}_{tts}$. □

Consequence 3.4. $\mathcal{LNDR}_{tts}=\mathcal{LDR}_{tts}$.

Proof. If $\mathfrak{A}$ in Lemma 3.3 is an LDR transducer then $\mathfrak{B}$ is an LNDR transducer. □

Consequence 3.5. $\mathcal{DR}_{tts}=(\mathcal{NH}\circ\mathcal{LNDR})_{tts}$.

Proof. It is well known that $\mathcal{DR}=\mathcal{NH}\circ\mathcal{LDR}$ (c.f. [1], [4]) thus we have $\mathcal{DR}_{tts}=(\mathcal{NH}\circ\mathcal{LDR})_{tts}=\mathcal{NH}\circ\mathcal{LDR}_{tts}=\mathcal{NH}\circ\mathcal{LNDR}_{tts}=(\mathcal{NH}\circ\mathcal{LNDR})_{tts}$. □

Now we are ready to state our last theorem.

Theorem 3.6. The poset $\langle \mathrm{yd}([S](\mathcal{R}ec)), \subseteq\rangle$ is a chain of three elements

$$\mathrm{yd}(\mathcal{R}ec)\subset \mathrm{yd}(\mathcal{NH}(\mathcal{R}ec))\subset \mathrm{yd}(\mathcal{DR}(\mathcal{R}ec)).$$

Proof. By Consequence 3.5, we can compute as follows:

$\mathrm{yd}(\mathcal{NH}\circ\mathcal{C}_0(\mathcal{R}ec))=\mathrm{yd}(\mathcal{NH}\circ\mathcal{LNDR}(\mathcal{R}ec))=(\mathcal{NH}\circ\mathcal{LNDR})_{tts}(\mathcal{R}ec)=\mathcal{DR}_{tts}(\mathcal{R}ec)=$ $=\mathrm{yd}(\mathcal{DR}(\mathcal{R}ec))$. Thus applying yd to each element of $\langle[S](\mathcal{R}ec), \subseteq\rangle$ we obtain the chain $\mathrm{yd}(\mathcal{R}ec)\subseteq\mathrm{yd}(\mathcal{NH}(\mathcal{R}ec))\subseteq\mathrm{yd}(\mathcal{DR}(\mathcal{R}ec))$. Here each inclusion is proper as it was shown in [2]. □

Finally we have the consequence mentioned before.

Consequence 3.7. $\mathcal{NH}(\mathcal{R}ec)\subset\mathcal{NH}\circ\mathcal{C}_0(\mathcal{R}ec)$.

Proof. It is obvious since, by the proof of Theorem 3.6, the same proper inclusion holds for the yields of these two classes. □

RESEARCH GROUP ON THEORY OF AUTOMATA
HUNGARIAN ACADEMY OF SCIENCES
SOMOGYI U. 7., SZEGED, HUNGARY,
H—6720

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