A Decomposition Theorem for a Class of Infinite Transformation Semigroups

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1. Introduction

In recent years there has been a great deal of interest in infinite automata; see for example Bavel [2] and Gacs [9] (automata theory), Biermann [3] and Scott [15] (semantics), Reeker [11] (formal languages), and Reeker and Tucci [12] (algorithms). One approach to the study of infinite automata is to see how much of the theory of finite machines extends to infinite machines. The purpose of this paper is to generalize results on the decomposition of finite automata to infinite automata. Previous results on the decomposition of infinite automata include the results of Bavel [2], in which he decomposes an infinite automaton into the union of certain sub-automata. Rhodes [13], [14] and Warne [18], [19] have developed decompositions for infinite semigroups similar to the classical Krohn—Rhodes decomposition (Arbib [1]). Ésik [6], [7] and Ésik and Gécseg [8] have studied decomposition from the point of view of varieties. Tucci [16] has developed a wreath product decomposition of infinite automata in terms of reset machines, group machines, and a third type of machine known as unique predecessor machines (see Bavel [2]). This decomposition is in the spirit of the Krohn—Rhodes decomposition, although the decomposition itself is much weaker. It seems necessary to make certain assumptions on an infinite automaton to obtain a stronger decomposition, and that is what we do in this article.

In this paper we work w th transformation semigroups rather than with automata because the notation is simpler. We develop a decomposition theorem for a certain class of unique predecessor transformation semigroups (Bavel [2]). The basic idea is to generalize the holonomy decomposition theorem of Eilenberg [5, theorem 7.1] to infinite transformation semigroups. We choose the holonomy decomposition theorem because it generalizes to the infinite case in a fairly natural manner. We follow closely the exposition presented in Holcombe [10], especially in the last section of this paper.

The second section of this paper develops some technical results on the skeleton of a transformation semigroup, as defined in Holcombe [10], and the third section describes what we call the depth function, which is the dual of the height function given in Holcombe [10]. The fourth section describes the structure of the semigroups into which a certain class of transformation semigroups can be decomposed. The final section contains the main decomposition theorem.

A **transformation semigroup** is an ordered pair $T=(Q, S)$ where Q is a set and S is a semigroup, together with a partial product $Q\times S\to Q$ denoted by concatenation, such that

(1) $(qs_1)s_2 = q(s_1s_2)$ for all $q\in Q$ and $s_1, s_2\in S$;
(2) if $s_1, s_2\in S$ and $qs_1=qs_2$ for all $q\in Q$, then $s_1=s_2$.

Throughout this paper the symbol T always stands for the transformation semigroup (Q, S). We assume that Q contains more than one element, and that Q and S are countable. If A is any subset of Q, then $|A|$ denotes the size of A. The semigroup S is called the **abstract** (or **action**) **semigroup** of S. We will assume that S contains an identity 1 which satisfies the property that $q\cdot 1=q$ for all $q\in Q$. If S is generated by elements $s_1, s_2, ..., s_n, ...$, then we denote this by writing $S=\langle s_1, s_2, ..., ..., s_n, ...\rangle$.

For each $s\in S$ we let F_s be the **partial function induced** by s, where F_s is given by the rule $F_s(q)=F(q, s)$ for all $q\in Q$. Note that F_s is single-valued where it is defined, but that F_s may not be defined on all of Q; the set $\{q\in Q|qF_s$ is defined$\}$ is the **domain of** s, denoted dom s. Sometimes for convenience we write qF_s, or simply qs, for $F_s(q)$. If $a, b\in S$ and qF_aF_b is undefined for all $q\in Q$ (i.e., the domain of F_b is disjoint from the range of F_a), then we adjoin a zero 0 to S and define $ab=0$. We can think of 0 as inducing a partial function on Q whose domain is $\emptyset$. A transformation semigroup $T=(Q, S)$ is a **unique predecessor transformation semigroup** if F_s is a $1-1$ map for each $s\in S$ (Bavel, [2, p. 576]). When T is a unique predecessor transformation semigroup, we can define the set $S^{-1}=\{s^{-1}|s\in S\}$ where $F_s-1= =(F_s)^{-1}$, the partial function which is defined by the rule $F_s-1(q)=q'$ if and only if $F_s(q')=q$ for all $q, q'\in Q$. Note that dom $s=Qs^{-1}$ for all $s\in S$. We define the **quotient of** T as the transformation semigroup $T'=(Q, S\cup S^{-1})$. The transformation semigroups we consider in this paper are all quotients of unique predecessor transformation semigroups. Note that if $s=s_1s_2...s_n\in S$, then S contains the element $s^{-1}=s_n^{-1}...s_2^{-1}s_1^{-1}$, where $ss^{-1}s=s$. Hence the action semigroup of the quotient of a unique predecessor transformation semigroup is regular.

Throughout this paper the symbol N stands for the set of positive integers. We assume that the reader is familiar with the definitions of the restricted direct product and the wreath product of transformation semigroups as given in Holcombe [10]. We also assume that the reader is familiar with the basic theory of semigroups as presented in Clifford and Preston [4].

2. The skeleton

To begin we need some preliminary definitions.

2.1. Definition. The **skeleton** (Holcombe [10, p. 119]) of a transformation semigroup T, denoted $I(T)$, is the collection of subsets of Q of the form $\emptyset$, $\{q\}$ (where $q\in Q$), or Qs for any $s\in S$. Since S contains an identity element by definition, we have that $Q\in I(T)$. Also, as we have observed earlier, if $s\in S$, then dom $s=Qs^{-1}$, so that dom $s\in I(T)$ for every $s\in S$. If A, B are two skeleton elements, then $A\leqq B$ if there is some $s\in S$ such that $A\subseteqq Bs$. Since S contains an identity element, and since this identity induces the identity map on Q, the condition $A\subseteqq B$ implies that $A\leqq B$. Two skeleton elements A, B are **equivalent** (Holcombe [10, p. 119]), denoted $A\equiv B$, if $A\leqq B$ and $B\leqq A$; i.e., $A\equiv B$ if there are elements $s, t\in S$ such that $A\subseteqq Bs$ and $B\subseteqq At$. If $A\leqq B$ and $A\not\equiv B$, then we indicate this by writing $A\prec B$. The elements A, B are **strongly equivalent**, denoted $A\equiv_s B$, if there are $s, t\in S$ such that $A=Bs$, $B=At$.

In a finite transformation semigroup, if $A, B\in I(T)$, $A\subseteqq B$, and $A\equiv B$, then $A=B$. However, in an infinite transformation semigroup we can have $A, B\in I(T)$, $A\subseteqq B$, and $A\equiv B$ but $A\neq B$.

2.2. Example. Let $T=(Q, S)$ where $Q=\{q_n|n\in N\}$, $S=\langle s, s^{-1}\rangle$, and $F_s(q_n)=q_{n+1}$ for all $q\in Q$. The partial function F_s-1 is defined in the obvious manner. If we take $A=\{q_n|n>1\}=Qs$ and $B=Q$ then $A, B\in I(T)$ and $A\equiv B$ but $A\neq B$. ∎

In a finite transformation semigroup, equivalent is the same as strongly equivalent (Holcombe [10, proposition 4.2.2]). This is not necessarily the case in an infinite transformation semigroup.

2.3. Example. Let $T=(Q, S)$ where $Q=\{p_n|n\in N\}\cup\{q_n|n\in N\}$ and $S=\langle a, b, s, t, a^{-1}, b^{-1}, s^{-1}, t^{-1}\rangle$. Define

(1) $F_a(p_j)=p_j$ for all $j\geqq 2$;

(2) $F_b(q_j)=q_j$ for all $j\geqq 2$;

(3) $F_s(q_j)=p_{j-1}$ for all $j\geqq 2$;

(4) $F_t(p_j)=q_{j-1}$ for all $j\geqq 2$.

The partial functions induced by $a^{-1}, b^{-1}, c^{-1}, t^{-1}$ are defined in the obvious way. For any other $x\in S$ and $q\in Q$, we have that $F_x(q)$ is undefined.

Let $Qa=A=\{p_j|j\geqq 2\}$, and let $Qb=B=\{q_j|j\geqq 2\}$. Then $A\subseteqq Bs$ and $B\subseteqq At$ but there are no elements $s', t'\in S$ such that $A=Bs'$ and $B=At'$. ∎

We now develop a simple but important criterion which we need in section 5 to make our decomposition work.

2.4. Definition. The skeleton $I(T)$ of the transformation semigroup T **satisfies the weak ascending chain condition,** or *WACC,* if every ascending chain of non-equivalent skeleton elements under the relation $\leqq$ halts after finitely many steps. Similarly, the semigroup S **satisfies the ascending chain condition on some class C of left (right, two-sided) ideals** if every increasing chain under inclusion of left (right, two-sided) ideals from class C halts after finitely many steps.

2.5. Example. The transformation semigroup in figure 2.1 does not satisfy *WACC* on its skeleton. The state set of this transformation semigroup consists of infinitely many components, where each component contains one more vertical edge labeled b than does the previous component. This transformation semigroup has the chain $Qaba\subsetneqq Qab^2a\subsetneqq\dots\subsetneqq Qab^na\subsetneqq\dots$ since $Qaba=\{q_1\}$, $Qab^2a=\{q_1, q_2\}$, etc.

(For simplicity we omit edges labeled by a^{-1} and b^{-1}.) ∎

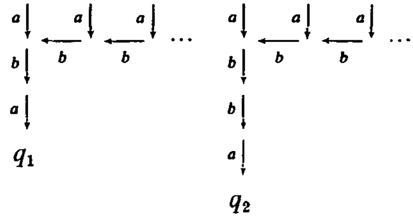

Figure 2.1

2.6. Proposition. Let T be a transformation semigroup, and let $Qa, Qb \in I(T)$. Then $Qa \cap Qb \in I(T)$.

Proof. It is easy to see that $Qa \cap Qb = Qab^{-1}b = Qba^{-1}a$. ∎

2.7. Proposition. The skeleton of the transformation semigroup $T=(Q, S)$ satisfies *WACC* if and only if S satisfies *ACC* on cyclic left ideals.

Proof. Let $A=Qa$, $B=Qb$ for some $a, b \in S$. Suppose first that $A \leqq B$. Then there is some $s \in S$ such that $Qa \subseteqq Qbs$. By proposition 2.6 $Qa = Qa \cap Qbs = Qa(bs)^{-1}bs$. It can be shown that $\operatorname{dom} a(bs)^{-1}bs = \operatorname{dom} a$, so that $a = a(bs)^{-1}bs$. Hence $SaS \subseteqq SbS$. Conversely, if $SaS \subseteqq SbS$, then $a = xby$ for some $x, y \in S$, so that $Qa = Qxby \subseteqq Qby$ and therefore $A \leqq B$. In a similar fashion, we can prove that $A \subseteqq B$ if and only if $Sa \subseteqq Sb$. Therefore, $A \equiv B$ if and only if $SaS = SbS$ or $Sa = Sb$. Hence, $I(T)$ satisfies *WACC* if and only if S satisfies *ACC* on cyclic left ideals and on cyclic two-sided ideals. The result now follows from the fact that every two-sided ideal is a left ideal. ∎

From now on, we assume that the skeletons of all transformation semigroups under consideration satisfy *WACC*. We conclude this section with a technical result.

2.8. Proposition. Let $T=(Q, S)$ be a transformation semigroup, and let $A, B \in I(T)$, where $A=Qa$, $B=Qb$ for some $a, b \in S$. For any $s \in S$, $(Qa \cap Qb)s = Qas \cap Qbs$.

Proof. Let $q \in (Qa \cap Qb)s$. Then $q \in Qas$ and $q \in Qbs$ so $q \in Qas \cap Qbs$. Conversely, suppose $q \in Qas \cap Qbs$. Then there exist $q_1, q_2 \in Q$ such that $q = (q_1 a)s = (q_2 b)s$. Since all elements of S induce $1-1$ maps on Q, we have $q_1 a = q_2 b$, so that $q_1 a \in (Qa \cap Qb)$. Thus, $q \in (Qa \cap Qb)s$. ∎

3. The depth function

3.1. Definition. Let $T=(Q, S)$ be a transformation semigroup. A **depth function** is a function d from $I(T)$ to the class of ordinals such that

(1) $d(Q)=0$.

(2) $d(\{q\}) > d(A)$ for all $q \in Q$ and $A \in I(T)$ with $|A|>1$, and $d(\emptyset) > d(\{q\})$ for all $q \in Q$.

(3) If $A \equiv B$ then $d(A)=d(B)$.
(4) If $A<B$ then $d(A)>d(B)$.
(5) If there are $A, B \in I(T)$ such that $d(A)=n$ and $d(B)=m$ then for every ordinal k with $n<k<m$ there is some $C \in I(T)$ such that $d(C)=k$.

The depth function as defined in definition 3.1 is the dual of the height function as given in Holcombe [10, p. 121]. To construct a depth function we need the following proposition.

3.2. Proposition. Let $T=(Q, S)$ be a transformation semigroup and let $A, B \in I(T)$. Define a function d from $I(T)$ to the class of ordinals as follows:
(1) $d(Q)=0$.
(2) Let $|A|>1$. If $A \equiv Q$ then $d(A)=0$; otherwise, $d(A)=\sup\{1+d(B) \mid A<B\}$.
(3) If $|A|=1$, then $d(A)=\sup\{1+d(B) \mid |B|>1\}$.
(4) If $A=\emptyset$ then $d(A)=1+d(B)$, where B is any one-element set in the skeleton.
Then d is a depth function. (The **depth of the transformation semigroup** is $d(\{q\})$ for any $q \in Q$, which we denote by $d(T)$.)

Proof. We need to show that the conditions of definition 3.1 are satisfied.
3.1 (1) This follows from the definition of d.
3.1 (2) This follows from the definition of d.
3.1 (3) Let $A \equiv B$. If $|A|=|B|=1$ or $A=B=\emptyset$ then $d(A)=d(B)$ by definition. It is not possible that A is empty or a singleton while B is not, because of the definition of equivalence.
Suppose now that $|A|>1$, $|B|>1$. Then for any $C \in I(T)$ we have $A<C$ if and only if $B<C$.
3.1 (4) If $A<B$, then by definition $d(A)>d(B)$.
3.1 (5) This follows from the definition of d. ∎

3.3. Proposition. Let $T=(Q, S)$ be a transformation semigroup, let $A \in I(T)$, and let d be a depth function.
(1) If $d(A)=0$ then $A \equiv Q$.
(2) If $A, B \in I(T)$ with $A \not\equiv B$, $A \subsetneqq B$, and $|B|>1$ then $d(A)>d(B)$.

Proof. These facts follows immediately from the definition of d. ∎

4. The holonomy transformation semigroup

In this section we describe the structure of the transformation semigroups which we use in our decomposition.

4.1. Definition. Let $T=(Q, S)$ be a transformation semigroup and let $G(S)$ be the group of units of S. For each $A \in I(T)$, let $S_1(A)=\{s \in S \mid As=A \subseteqq \operatorname{dom} s, s \notin G(S)\}$, and let $S(A)=S_1(A) \cup G(S)$. Take $J(S_1(A))$ to be the ideal generated by $S_1(A)$ (if $S_1(A)=\emptyset$ then define $J(S_1(A))=0$), and let $J(A)=\bigcup\{J(S_1(A')) \mid A' \subseteqq A, A' \equiv A\}$. The **holonomy transformation semigroup of** A denoted $T(A)$, has as its state set the set $\{Bx \mid B \in I(T), B \subseteqq A, B \not\equiv A$, and $x \in J(A) \cup G(S)\}$; the action semigroup of $T(A)$ is defined by taking $J(A) \cup G(S)$ and identifying those elements which act the same on the state set of $T(A)$. We will denote the state set of $T(A)$ by $ST(A)$ and the action semigroup of $T(A)$ by $X(A)$.

We define $T(A)$ as we do for several reasons. First of all, if the transformation semigroup is finite then definition 4.1 reduces to $S_1(A)=\{s|As=A\}$, and this is the definition of $S(A)$ in the finite case in Holcombe [10, p, 123]. (Note that $T(A)$ contains the set of all permutations of A onto itself.) Second, in the finite case $S(A)$ is a set of permutations on the maximal skeleton elements contained in A (Holcombe [10, proposition 4.3.1]); in the infinite case, the elements of $S_1(A)$ map the skeleton elements in A to skeleton elements of A of the same depth in a $1-1$ fashion (proposition 4.5). Third, we define $ST(A)$ as we do because there are some elements in $X(A)$ which induce maps on A that do not necessarily preserve the depth of skeleton elements in A, and in fact may not map into A at all (proposition 4.5, example 4.6). Finally, we define $T(A)$ by using $J(A)\cup G(S)$ to assure us that $X(A)$ has at least as much structure as S (proposition 4.2 (1) and (2)) and to assure us that one important technical detail works out (proposition 4.2 (3)).

4.2. Proposition. Let $T(A)=(ST(A), X(A))$ be the holonomy semigroup of A.
(1) $X(A)$ is regular.
(2) $X(A)$ satisfies ACC on cyclic left ideals.
(3) If $As\equiv A$ for some $s\in S$ then s embeds in $X(A)$.

Proof. (1) The ideal $J(A)$ is a regular semigroup, for if $s\in J(A)$, then $s^{-1}=s^{-1}ss^{-1}\in J(A)$. Further, $J(A)\cup G(S)$ is a regular semigroup, and $X(A)$ is a homomorphic image of $J(A)\cup G(S)$.

(2) Any cyclic left ideal of $J(A)$ is of the form $J(A)a\cup\{a\}$ where $a\in J(A)$. But $J(A)a$ is also a left ideal of S. Hence $J(A)$ satisfies ACC on cyclic left ideals. Since every left ideal in $J(A)\cup G(S)$ is contained in $J(A)$, we have that $J(A)\cup G(S)$ satisfies ACC on cyclic left ideals. The result now follows because $X(A)$ is a homomorphic image of $J(A)\cup G(S)$.

(3) Suppose $As\equiv A$. If $s\in G(S)$, then $s\in T(A)$ by definition. If $s\notin G(S)$, let $K=A\cap \operatorname{dom} s$. By proposition 2.6 $K\in I(T)$. Also, $A\equiv As=Ks\leqq K\leqq A$ and hence $K\equiv A$. Further, $Kss^{-1}=K$ and $K\subseteqq \operatorname{dom} ss^{-1}$, so that $ss^{-1}\in S_1(K)\subseteqq J(K)\subseteqq J(A)$. But $J(A)$ is an ideal, so that $s=ss^{-1}s\in J(A)$. Hence s embeds in $X(A)$. ∎

Although $ST(A)$ consists of more than just the skeleton elements of T in A, we concentrate our attention mostly on these latter elements. To study these skeleton elements we need the following notation.

4.3. Definition. Let T be a transformation semigroup and let $A', A\in I(T)$, $A'\subseteqq A$. If $A\equiv A'$ then we define $d_A(A')=0$. If $A\not\equiv A'$ then $d_A(A')=\sup\{1+d_A(B)|B\in I(T), A'\subseteqq B\subseteqq A, A'\not\equiv B\}$. Note that for the function d_A we take chains under inclusion, not chains under the relation $\leqq$. Note also that $d_A(A')$ may be an infinite ordinal. When there is no ambiguity we abbreviate $d_A(A')$ by $d(A')$. Further, we define

$$I(A) = \{A'\in I(T)|A' \subseteqq A\}$$
$$I_n(A) = \{A'\in I(T)|A' \subseteqq A,\, d_A(A') = n\}$$
$$I_{n^+}(A) = \{A'\in I(T)|A' \subseteqq A,\, d_A(A') \geqq n\}$$
$$I_{n'}(A) = \{A'\in I(T)|A' \subseteqq A,\, d_A(A') > n\}.$$

4.4. Lemma. Let $T=(Q, S)$ be a transformation semigroup. If $A'\in I(A)$ and $Ax\subseteq B$ for some $x\in S$ where $A\subseteq \operatorname{dom} x$ and $B\in I(T)$, then $d_A(A')\leqq d_B(A'x)$.

Proof. Since $A\subseteq \operatorname{dom} x$ we know that the map $C\to Cx$ for every $C\in I(A)$ is an isomorphism between $I(A)$ and $I(Ax)$ as posets ordered by inclusion. Moreover, $C\leqq D$ if and only if $Cx\leqq Dx$, for all $C, D\in I(A)$. But then $d_A(A')=d_{Ax}(A'x)\leqq \leqq d_B(A'x)$ follows. This completes the proof. ∎

4.5. Proposition. Let $T=(Q, S)$ be a transformation semigroup and let $A\in I(T)$.

(1) Let $s\in S_1(A)$, and let $B\in I_n(A)$. Then the map $B\to Bs$ is a $1-1$ onto map from $I_n(A)$ to $I_n(A)$.

(2) If $s\in S$ and s induces a $1-1$ map from $I_n(A)$ to $I_n(A)$ for all n, then s embeds in $X(A)$.

Proof. (1) It is obvious that each $s\in S_1(A)$ maps $I(A)$ into $I(A)$. We first show that if $B\in I_n(A)$ then $Bs\in I_n(A)$. If $Bs=B$ then we are done. Otherwise, since $As=A\subseteq \operatorname{dom} s$, we may directly infer from lemma 4.4 that $d_A(B)\leqq d_A(Bs)$. Now $A=As\subseteq \operatorname{dom} s$ implies that $As^{-1}=Ass^{-1}=A$, and $A=As$ implies that $A\subseteq \operatorname{dom} s^{-1}$, so that again by lemma 4.4 we have that $d_A(Bs)\leqq d_A(Bss^{-1})$. Since $Bss^{-1}=B$, we have that $d_A(B)=d_A(Bs)$, so $Bs\in I_n(A)$. (Note we have proven that if $s\in S_1(A)$ then $s^{-1}\in S_1(A)$.)

To show that the map $B\to Bs$ is onto, note that $B=(Bs^{-1})s$ where the argument of the previous paragraph shows that $Bs^{-1}\in I_n(A)$. To show that the map is $1-1$, suppose that $Bs=B's$ for some $B'\in I_n(A)$. Then $B=B'$ because every element of S is $1-1$ on its domain.

(2) Let s be any element of S which maps $I_n(A)$ to $I_n(A)$ in a $1-1$ onto fashion. If $s\in G(S)$, then automatically s embeds in $X(A)$. If $s\notin G(S)$, then by hypothesis the element s maps $I_0(A)$ to $I_0(A)$. In particular $As\equiv A$. By proposition 4.2, s embeds in $X(A)$. ∎

In the finite case the condition $As\subseteq A\subseteq \operatorname{dom} s$ implies that $As=A$, and hence that s permutes the elements of $I(A)$. This is not necessarily true in the infinite case; in particular, we may have that $As^{-1}\not\subseteq A$.

4.6. Example. Let $T=(Q, S)$ be a transformation semigroup where $Q= =\{q_n|n\in N\}$ and $S=\langle a, s, a^{-1}, s^{-1}\rangle$. Define:

(1) $F_a(q_n)=q_n, \quad n\geqq 2;$

(2) $F_s(q_n)=q_{n+1}, \quad n\geqq 1.$

The partial functions induced by a^{-1} and s^{-1} are defined in the obvious manner. For any other $x\in S$ and $q\in Q$, we have that $F_x(q)$ is undefined.

The set $A=Qa=\{q_n|n\geqq 2\}$ is a skeleton element of this transformation semigroup, and $As\subseteq A\subseteq \operatorname{dom} s$. However, $q_1\in As^{-1}$ and $q_1\notin A$. ∎

If we assume that $As\subseteq A\subseteq \operatorname{dom} s$ and if we also assume that $As^{-1}\subseteq A$, then s^{-1} may still not map $I_n(A)$ to $I_n(A)$ for all n.

4.7. Example. Let $T=(Q, S)$ be a transformation semigroup where $Q= =\{q_n|n\in N\}$ and $S=\langle s, t, s^{-1}, t^{-1}\rangle$. Define:

(1) $F_s(q_n) = q_{n+1}, \quad n \geqq 1;$

(2) $F_t(q_1) = q_1$ and $F_t(q_2) = q_2$.

The partial functions induced by s^{-1} and t^{-1} are defined in the obvious manner. For any other $x \in S$ and $q \in Q$, we have that $F_x(q)$ is undefined.

Let $A=Q$, and note that $As \subseteqq A \subseteqq \operatorname{dom} s$ (in fact, $A = \operatorname{dom} s$) and $As^{-1} \subseteqq A$. Then $B=Qt=\{q_1, q_2\} \in I(A)$ and $Bs^{-1}=\{q_1\}$. Since $Bs^{-1} \subseteqq B$ but $Bs^{-1} \not\equiv B$, we have $d_A(Bs^{-1}) > d_A(B)$. In particular, s^{-1} does not map $I_n(Q)$ to $I_n(Q)$ for all n. ▮

We conclude this section by describing how the relation $A \equiv B$ induces a relation between $I(A)$ and $I(B)$.

4.8. Proposition. Let $T=(Q, S)$ be a transformation semigroup, and let $A, B \in I(T)$ where $A \equiv B$ with $A \subseteqq Bs$ and $B \subseteqq At$ for some $s, t \in S$.

(1) $I_n(A) \subseteqq I_{n^+}(B)s$ and $I_n(B) \subseteqq I_{n^+}(A)t$ for all n.

(2) If $A=Bs$ and $B=At$, then $I_n(A) \subseteqq I_n(B)s$ and $I_n(B) \subseteqq I_n(A)t$ for all n.

Proof. (1) Let $A' \in I_n(A)$, and let $B'=\{b \in B \mid bs=a$ for some $a \in A'\}$. Note that $B'=A's^{-1}$, so that $B' \in I(B) \subseteqq I(T)$. Since $A \subseteqq Bs$, we have $A'=B's$. Suppose that $B' \in I_m(B)$. We must show that $m \geqq n$.

Choose any chain of non-equivalent skeleton elements from A to A', say

$$A \supset A_1 \supset \ldots \supset A'.$$

Since $A \subseteqq Bs$ we have that $A_j \subseteqq \operatorname{dom} s^{-1}$ for all A_j. Therefore

$$B \equiv As^{-1} \supseteqq A_1 s^{-1} \supseteqq \ldots \supseteqq A's^{-1} = B'.$$

By the argument of lemma 4.4, the elements of this chain are all non-equivalent. Therefore $m \geqq n$.

(2) Suppose now that $A=Bs$ and $B=At$. We prove that $m \leqq n$. Choose any chain of non-equivalent skeleton elements from B to B', say

$$B \supset B_1 \supset \ldots \supset B'.$$

Since $B \subseteqq At$, for each $B_j \subseteqq B$ there exists some $A_j \subseteqq A$ such that $B_j = A_j t$, and there is also some $A'' \subseteqq A$ such that $B'=A''t$; that is, $A_j = B_j t^{-1}$ and $A''=B't^{-1}$. Thus, we can rewrite this chain as

$$B \supset A_1 t \supset \ldots \supset A''t = B'$$

This gives rise to a chain in A, namely

$$A \equiv Bt^{-1} \supseteqq A_1 \supseteqq \ldots \supseteqq A''.$$

But then $A'=B's=A''ts$ and $A=Bs=Ats$ which yields the chain

$$A = Ats \supseteqq A_1 ts \supseteqq \ldots \supseteqq A''ts = A'.$$

Again, by the argument of lemma 4.4, the elements of this chain are all non-equivalent. Therefore $m \leqq n$. It follows that $m=n$. ▮

The containments $I_n(A) \subseteqq I_{n^+}(B)s$ and $I_n(B) \subseteqq I_{n^+}(A)t$ in proposition 4.8 (1) cannot be replaced by $I_n(A) \subseteqq I_n(B)s$ and $I_n(B) \subseteqq I_n(A)t$.

4.9. Example. Let $T=(Q, S)$ be a transformation semigroup where

$$Q = \{p_n | n \in N\} \cup \{q_n | n \in N\}$$

and

$$S = \langle a, a', b, b', s, t, a^{-1}, a'^{-1}, b^{-1}, b'^{-1}, s^{-1}, t^{-1} \rangle.$$

Define:

(1) $F_a(p_n) = p_n$ for all $n \geqq 2$;

(2) $F_{a'}(p_n) = p_n$ for all $n \geqq 3$;

(3) $F_b(q_n) = q_n$ for all $n \geqq 2$;

(4) $F_{b'}(q_n) = q_n$ for all $n \geqq 3$;

(5) $F_s(q_n) = p_{n-1}$ for all $n \geqq 2$;

(6) $F_t(p_n) = q_{n-1}$ for all $n \geqq 2$.

The partial functions induced by a^{-1}, a'^{-1}, b^{-1}, b'^{-1}, s^{-1}, t^{-1} are defined in the obvious manner. For all other $x \in S$ and $q \in Q$, we have that $F_x(q)$ is undefined.

Let $A=Qa$ and let $B=Qb$. Then $A \subseteqq Bs$ and $B \subseteqq At$. By definition, $A \in I_0(A)$. If we let $B_1=Qb'=\{q_n | n \geqq 3\}$ then $A=B_1 s$ but $B_1 \notin I_0(B)$ because $B_1 \not\equiv B$. Note also that $A \neq B's$ for any other $B' \in I(B)$. ▮

The containments $I_n(A) \subseteqq I_n(B)s$ and $I_n(B) \subseteqq I_n(A)t$ in proposition 4.8 (2) cannot be replaced with equalities.

4.10. Example. Let $T=(Q, S)$ be a transformation semigroup where

$$Q = \{p_n | n \in N\} \cup \{q_n | n \in N\}$$

and

$$S = \langle a, b, c, x, s, t, a^{-1}, b^{-1}, c^{-1}, x^{-1}, s^{-1}, t^{-1} \rangle.$$

Define:

(1) $F_a(p_j) = p_j$ for $j \geqq 1$;

(2) $F_b(q_j) = q_j$ for $j \geqq 1$;

(3) $F_c(p_j) = p_{(j/2)}$ if j is even;

(4) $F_x(p_j) = p_{j+1}$ if j is odd;

(5) $F_s(q_j) = p_j$ for all $j \geqq 1$;

(6) $F_t(p_2) = q_1$, $F_t(p_{2j+1}) = q_{j+1}$ for $j \geqq 1$.

The partial functions induced by a^{-1}, b^{-1}, c^{-1}, s^{-1}, t^{-1}, and x^{-1} are defined in the obvious manner. For all other $y \in S$ and $q \in Q$, the expression $F_y(q)$ is undefined.

The sets $A=Qa=\{p_j | j \geqq 1\}$ and $B=Qb=\{q_j | j \geqq 1\}$ are strongly equivalent skeleton elements, with $A=Bs$ and $B=At$. Let $X=Ax=\{p_{2j} | j \geqq 1\}$. Then $X \subseteqq A$, $A \subseteqq Xc$, so that $X \in I_0(A)$ but $Xt=\{q_1\} \neq B$, so that $Xt \notin I_0(B)$. ▮

5. The decomposition theorem

We now develop the decomposition theorem. The basic idea for this decomposition is the same as in the finite case; we start off with a "coarse" decomposition and refine it until we get the result we desire. Throughout this section we follow closely the presentation in Holcombe [10, chapter 4].

5.1. Definition. (1) (Holcombe [10, p. 102]). Let $T=(Q, S)$ be a transformation semigroup. Let $\pi=\{H_j\}_{j\in I}$ be a collection of subsets of Q such that $Q=\bigcup_{j\in I}H_j$, where I is some indexing set for this collection. Then π is an **admissible subset system** if for any $i\in I$ and $s\in S$ there exists $j\in I$ such that $H_iF_s\subseteq H_j$.

(2) Let π, π' be two admissible subset systems. Then we say that $\pi'\leqq\pi$ if for every $H'\in\pi'$ there is some $H\in\pi$ such that $H'\subseteq H$.

5.2. Definition. Let $T=(Q, S)$ and $T'=(Q', S')$ be two transformation semigroups.

(1) (Holcombe [10, p. 43]). A partial function $\alpha\colon Q'\to Q$ is a **covering of T by T'** if

(a) α is surjective;

(b) for every $s\in S$ there is some $t_s\in S'$ such that either $\alpha(q')s$ is undefined or $\alpha(q')s=\alpha(q't_s)$ for every $q'\in Q'$.

We denote the fact that T' covers T by writing $T\leqq T'$.

(2) (Holcombe [10, p. 116]). A relation α on $Q'\times Q$ is called a **relational covering of T by T'** if

(a) α is surjective;

(b) for every $s\in S$ there is some $t_s\in S'$ such that $\alpha(q')s\subseteq\alpha(q't_s)$ for every $q'\in Q'$.

We denote the fact that α is a relational covering of T by T' by writing $T\leqq_\alpha T'$.

5.3. Definition. (Holcombe [10, p. 122]). Let $T=(Q, S)$ and $T'=(Q', S')$ be two transformation semigroups, where T has depth function d. Let α be a relational covering of T by T'. Then α has **rank** i (with respect to d) if

(1) $\alpha(q')\in I(T)$ for all $q'\in Q'$;

(2) $d(\alpha(q'))\geqq i$ for all $q'\in Q'$ and $d((\alpha(q'))=i$ for at least one $q'\in Q'$ where $0\leqq i\leqq d(T)$.

5.4. Definition. (Holcombe [10, p. 35]). Let $T=(Q, S)$ be a transformation semigroup. The **closure of** S is the set $\bar{\mathbf{S}}=S\cup\{\bar{\mathbf{q}}|q\in Q\}$ where, for each $q\in Q$, $\bar{\mathbf{q}}$ is the constant map defined by $x\bar{\mathbf{q}}=q$ for all $x\in Q$. The **closure of** T is $\bar{\mathbf{T}}=(Q, \bar{\mathbf{S}})$.

For each ordinal j, $0\leqq j\leqq d(T)$, we divide the set of skeleton elements of T at depth j into equivalence classes under the relation $\equiv$, and we take a set of representatives from these classes, say $A_1^j, A_2^j, \dots$. We form the holonomy transformation semigroups for all A_k^j and take their join $T(A_1^j)\vee T(A_2^j)\vee\dots$. This is denoted by $T_j^\vee(T)$, a transformation semigroup with state set denoted by $ST_j^\vee(T)$ and action semigroup denoted by $X_j^\vee(T)$. Note that the sets at depth 0 are all equivalent to Q, so we can choose $A_1^0=Q$, and hence $T_0^\vee(T)=T(Q)$. To ensure that the state sets of the $T(A_k^j)$'s are disjoint we will consider the state set of $T(A_k^j)$ to be $\{k\}\times ST(A_k^j)$ instead of $ST(A_k^j)$. Thus, a typical element of the state set of $T(A_k^j)$ is denoted (k, B_k^j) where $k\geqq 1$, $B_k^j\in ST(A_k^j)$.

The next definition generalizes a definition in Holcombe [10, p. 126].

5.5. Definition. For a transformation semigroup $T=(Q, S)$, the set π^j is $\{A\in I(T)|d(A)\geqq j\}$.

Note that π^j is an admissible subset system. We have $Q=\bigcup_{H\in\pi} jH$ because if $j\leqq d(T)$ then $d(\{q\})\geqq j$ for all $q\in Q$ and so $\{q\}\in\pi^j$. Also, if $B\in\pi^j$ and $s\in S$, then $d(Bs)\geqq d(B)\geqq j$ so that $Bs\in\pi^j$.

5.6. Theorem. Let $T=(Q, S)$ be a transformation semigroup of depth at least 1. Then there is a relational covering $T\leqq_\alpha \mathbf{T}_0^\vee(T)$ of rank 1.

Proof. By the argument of the previous paragraph, the set π^1 is an admissible subset system. To specify the covering, let $B'\in\pi^1$, let $s\in S$, and define

$$B' * s = \begin{cases} B's & \text{if } s\in T_0^\vee(T) \\ Qs & \text{otherwise.} \end{cases}$$

By proposition 4.2, if $s\notin T_0^\vee(T)$ then $Qs\not\equiv Q$, so that Qs is of depth 1 or greater. The pair (π^1, S) gives rise to the transformation semigroup $(\pi^1, S/\sim)$ where the congruence $\sim$ identifies any two elements of S which act identically on π^1. Denote $(\pi^1, S/\sim)$ by $T/\langle\pi^1\rangle$. If we define a relation $\alpha\colon \pi^1\to Q$ by $\alpha(B)=B$ for all $B\in\pi^1$ then we obtain a relational covering $T\leqq_\alpha T/\langle\pi^1\rangle$ of rank 1. We can in turn cover $T/\langle\pi^1\rangle$ by $\mathbf{T}_0^\vee(T)$. ∎

The proof of theorem 5.7 follows Holcombe [10, theorem 4.3.4].

5.7. Theorem. Let $T=(Q, S)$ and let d be a depth function. Let π be an admissible partition of rank j, where $j<d(T)$. Then there is an admissible subset system π' of rank $j+1$ with $\pi'\leqq\pi$.

Proof. Let $I_j(\pi)=\{A\in\pi|d(A)=j\}$, let $I_{j+}(\pi)=\{A\in\pi|d(A)>j\}$, and $I_{j++}(\pi)= =\{A\in I(T)|A\in I_{1+}(Y)$ for some $Y\in I_j(\pi)\}$. Define $\pi'=I_{j+}(\pi)\cup I_{j++}(\pi)$. Then $\pi'\leqq\pi$ and $\operatorname{rank}(\pi')=j+1$. We must show that π' is admissible.

We first show that $Q=\bigcup_{H\in\pi'} H$. Let $q\in Q$. If $q\in A\in L_{j+}(\pi)$ then there is nothing more to prove. If $q\notin A$ for any $A\in I_{1+}(\pi)$, then $q\in A$ for some $A\in I_j(\pi)$. But $d(\{q\})>j$ by definition so $\{q\}\in I_{j++}(\pi)$ by definition of $I_{j++}(\pi)$. Hence $Q=\bigcup_{H\in\pi'} H$.

Now let $B\in\pi'$, $s\in S$. We must show that $Bs\subseteqq A$ for some $A\in\pi'$. There are two main cases to consider.

Case I: $B\in I_{j+}(\pi)$. Then $B\in\pi$ so $Bs\subseteqq A$ for some $A\in\pi$ because π is admissible. There are two subcases to examine.

Subcase 1: $A\in I_{j+}(\pi)$. Then $Bs\subseteqq A\in\pi'$.

Subcase 2: $A\in I_j(\pi)$. Since $Bs\leqq B$ and $d(B)>j$ we have $d(Bs)>j$ and so $Bs\in I_{1+}(A)$ for $A\in I_j(\pi)$. Therefore Bs is an element of π'.

Case II: $B\in I_{j++}(\pi)$. Then $B\in I_{1+}(Y)$ for some $Y\in I_j(\pi)$ and $Bs\subseteqq Ys$. There are two subcases to consider.

Subcase 1: $Ys\subseteqq A$ for some $A\in I_j(\pi)$. Then $Bs\subseteqq A\in\pi'$.

Subcase 2: $Ys\subseteqq A$ for some $A\in I_j(\pi)$. Then $Bs\subseteqq Ys\subseteqq A$ and $d(Bs)\geqq d(B)>j$ so that $Bs\in I_{1+}(A)$. Hence $Bs\in\pi'$. ∎

5.8. Theorem. Let $T=(Q,S)$ and $T'=(Q',S')$ be transformation semigroups, and let $T \leqq_\alpha T'$ be a relational covering of rank j, $j<d(T)$, such that the image of α is π^j. Then there exists a relational covering $T \leqq_{\alpha'} \bar{\mathbf{T}}_j^{\vee}(T) \circ T'$ such that

(1) the rank of α' is $j+1$;

(2) the image of α' is π^{j+1}.

Proof. Since $T \leqq_\alpha T'$, for every $s \in S$ there is some $t_s \in S'$ such that $\alpha(q')s=\alpha(q't_s)$ for all $q' \in Q'$. Let $A_1^j, \ldots, A_k^j, \ldots$ be a set of representatives of equivalence classes under $\equiv$ of skeleton elements in T of depth j. Recall that $ST_j^{\vee}(T)=\bigcup_{1 \leqq k}(\{k\} \times ST(A_k^j))$ denotes the state set of $\bar{\mathbf{T}}_j^{\vee}(T)$ and that $X_j^{\vee}(T)$ denotes the action semigroup of $\bar{\mathbf{T}}_j^{\vee}(T)$.

To define the relation α' from $ST_j^{\vee}(T) \times Q'$ to Q consider an element $((k, B_k^j), q') \in K \times Q'$. If $d(\alpha(q'))=j$ then there is some A_k^j such that $\alpha(q') \equiv A_k^j$; in particular, $\alpha(q') \subseteqq A_k^j x$ for some $x \in S$. We define

$$\alpha'((k, B_k^j), q') = \begin{cases} \alpha(q') & \text{if } \ d(\alpha(q')) > j \\ B_k^j x \cap \alpha(q') & \text{if } \ \alpha(q') \equiv A_k^j, \quad \alpha(q') \subseteqq A_k^j x \\ \emptyset & \text{otherwise.} \end{cases}$$

By proposition 2.6, the image of α' is a skeleton element in all cases. Clearly, the rank of α' is greater than j.

We now show that the image of α' is π^{j+1}. Writing π^j as $I_j(\pi) \cup I_{j+}(\pi)$ as in theorem 5.7, we have $\pi^{j+1}=I_{j+}(\pi) \cup I_{j++}(\pi)$. Since the image of α is π^j, suppose that $Z \in \pi^{j+1}$. If $Z \in \pi^j$ then we have $d(Z)>j$ and $Z=\alpha(q')$ for some $q' \in Q'$ and so $\alpha'((k,B_k^j), q')=\alpha(q')=Z$ for any $(k,B_k^j) \in K$. If $Z \in I_{1'}(Y)$ for some $Y \in I_j(\pi)$ then $Y \in \pi^j$ and $Y=\alpha(q')$ for some $q' \in Q'$. Now $Y=\alpha(q') \equiv A_k^j$ for some $1 \leqq k$, so that $Y \subseteqq A_k^j x$ for some $x \in S$. Then $Z=B_k^j x$ for some $B_k^j \in I_{1+}(A_k)$ by proposition 4.8, and also $Z \subseteqq Y=\alpha(q')$. Therefore $Z=B_k^j x \cap \alpha(q')=\alpha'((k, B_k^j), q')$. Hence the image of α' equals π^{j+1}.

We now prove that α' is a relational covering. The crucial part is the definition of the element of the action semigroup $\bar{\mathbf{T}}_j^{\vee}(T) \circ T'$ which covers a given element of S.

Let $s \in S$ and suppose that t_s covers s with respect to the relational covering α. Thus $\alpha(q')s \subseteqq \alpha(q't_s)$ for all $q' \in Q'$. As before, $\bar{\mathbf{T}}_j^{\vee}(T)$ denotes the closure of the join of all the $T(A_k^j)$ for $k \geqq 1$. Now the action semigroup of $\bar{\mathbf{T}}_j^{\vee}(T) \circ T'$ consists of all ordered pairs (f, t) where $t \in S'$ and $f\colon Q' \to X_j^{\vee}(T)$. Having chosen our element $s \in S$ we define a function $f_s\colon Q' \to X_j^{\vee}(T)$ in the following way. Let $q' \in Q'$. Three possibilities arise:

Case I: $\alpha(q't_s) \in I_{j+}(\pi)$. Then f_s is chosen arbitrarily.

Case II: $\alpha(q't_s) \in I_j(\pi)$ and $\alpha(q')s \not\equiv \alpha(q't_s)$. Then $\alpha(q't_s) \equiv A_k^j$ for some $k \geqq 1$, so there is some $y \in S$ such that $\alpha(q't_s) \subseteqq A_k^j y$, so that $\alpha(q't_s)y^{-1} \subseteqq A_k^j$. Now $\alpha(q')sy^{-1} \leqq \alpha(q')s \not\equiv A_k^j$ and so $\alpha(q')sy^{-1} \subseteqq B'$ for some $B' \in I_{1+}(A_k^j)$. We put $f_s=C(B')$, the constant map which maps everything to B'.

Case III: $\alpha(q't_s) \in I_j(\pi)$ and $\alpha(q')s \equiv \alpha(q't_s)$. Then $\alpha(q') \equiv \alpha(q't_s)$ since $\alpha(q')s \leqq \alpha(q')$ and yet $\alpha(q')$ is of depth at least j. Now, as stated in the definition of

α', $A_k^j \equiv \alpha(q')$ implies that there is some $x \in S$ such that $\alpha(q') \subseteqq A_k^j x$. Thus $A_k^j xs \supseteqq \supseteqq \alpha(q')s \equiv \alpha(q't_s)$.

Now $\alpha(q')s \equiv \alpha(q't_s)$ implies that $\alpha(q')s \subseteqq \alpha(q't_s)x'$ for some $x' \in S$, so that in particular $\alpha(q')s(x')^{-1} \equiv \alpha(q')s$. Therefore $A_k^j xs(x')^{-1} \supseteqq \alpha(q')s(x')^{-1} \equiv \alpha(q')s \equiv \alpha(q') \equiv \equiv A_k^j$ which implies that $A_k^j xs(x')^{-1} \equiv A_k^j$. It follows from proposition 4.2 that $xs(x')^{-1} \in T(A_k^j) \subseteqq T_j^{\vee}(T)$, so we put

$$f_s(q') = xs(x')^{-1}.$$

This defines the function f_s: $Q' \to X_j^{\vee}(T)$. What remains is the task of showing that (f_s, t_s) covers s with respect to α'. Let $((l, B_l^j), q') \in K \times Q'$. We prove that

$$\alpha'((l, B_l^j), q')s \subseteqq \alpha'(((l, B_l^j), q')(f_s, t_s)). \tag{*}$$

Case I: $\alpha(q't_s) \in I_{j+}(\pi)$. If $\alpha(q') \in I_{j+}(\pi)$ then

$$\alpha'((l, B_l^j), q')s = \alpha(q')s \subseteqq \alpha(q't_s).$$

In all other cases

$$\alpha'((l, B_l^j), q')s \subseteqq \alpha(q')s \subseteqq \alpha(q't_s).$$

Since $\alpha(q't_s) \in I_{j+}(\pi)$, we have that $f_s(q')$ is arbitrary and that

$$\alpha'(((l, B_l^j), q')(f_s, t_s)) = \alpha'((l, B_l^j) f_s(q'), q't_s) = \alpha(q't_s).$$

Therefore the inequality (*) holds in this case.

Case II: $\alpha(q't_s) \in I_j(\pi)$ and $\alpha(q')s \not\equiv \alpha(q't_s)$. Now $f_s(q') = C(B')$, where $B' \in I_{1+}(A_k^j)$ and $\alpha(q')sy^{-1} \subseteqq B'$, where y is defined in Case II above to be the element such that $\alpha(q')s \subseteqq \alpha(q't_s) \subseteqq A_k^j y$. Therefore

$$\alpha'(((l, B_l^j), q')(f_s, t_s)) = \alpha((k, B'), q't_s) = B'y \cap \alpha(q't_s)$$

By definition of α', we have $\alpha'((l, B_l^j), q') \subseteqq \alpha(q')$, so that

$$\alpha'((l, B_l^j), q')s \subseteqq \alpha(q')s = \alpha(q')sy^{-1}y \subseteqq B'y \cap \alpha(q't_s)$$

and so (*) holds again.

Case III: $\alpha(q't_s) \in I_j(\pi)$ and $\alpha(q')s \equiv \alpha(q't_s)$. If $\alpha(q') \equiv A_k^j$ then

$$\alpha'(((l, B_l^j), q')(f_s, t_s)) = \alpha'((l, B_l^j xs(x')^{-1}, q't_s) = B_l^j xs(x')^{-1}x' \cap \alpha(q't_s).$$

Recall that x is the element of S for which $\alpha(q') \subseteqq A_k^j x$ and that x' is the element of S for which $\alpha(q't_s) \subseteqq A_k^j x'$; this latter inequality implies that $\alpha(q't_s) \subseteqq \mathrm{dom}\,(x')^{-1}$. In particular, $\alpha(q't_s)(x')^{-1}x' = \alpha(q't_s)$. Using these facts together with proposition 2.8 we can rewrite the last expression as

$$\begin{aligned} B_l^j xs(x')^{-1}x' \cap \alpha(q't_s)(x')^{-1}x' &= (B_l^j xs \cap \alpha(q't_s))(x')^{-1}x' = \\ &= B_l^j xs \cap \alpha(q't_s) \supseteqq B_l^j xs \cap \alpha(q')s = \\ &= (B_l^j x \cap \alpha(q'))s = \alpha'((1, B_l^j), q')s \end{aligned}$$

and so (*) holds. Finally, if $\alpha(q') \not\equiv A_l^j$ then $\alpha'((1, B_l^j), q') = \emptyset$ and so

$$\alpha'((1, B_l^j), q')s \subseteqq \alpha'(((1, B_l^j), q')(f_s, t_s))$$

as required. ▮

5.9. Main Decomposition Theorem. Let $T=(Q, S)$ be a transformation semigroup which is the quotient of a unique predecessor transformation semigroup, and let S satisfy ACC on cyclic left ideals. For each ordinal j, $0\leqq j\leqq d(T)$, let $A_1^j, \ldots, A_k^j, \ldots$ be a set of representatives of equivalence classes under $\equiv$ of skeleton elements of T of depth j. Then T is covered by a wreath product of transformation semigroups which are of one of the two following forms:

(1) (A_k^j, C_k^j) where C_k^j is the set of all constant maps from A_k^j to itself;

(2) $T_0^{\vee}(Q) = T(A_1^j)\vee \ldots \vee T(A_k^j)\vee \ldots$.

Further, if A stands for any A_k^j, then $T(A)=(ST(A), X(A))$, where

(a) $ST(A)=\{B|B\in I(T);\ B=B'x$ for some $B'\subseteqq A$, $B\not\equiv A$, and some $x\in X(A)\}$;

(b) $X(A)=(J(A)\cup G(S))/\sim$, where

(i) $J(A)$ is the ideal of S generated by the elements of S which induce a permutation on some $B\subseteqq A$, $B\equiv A$;

(ii) $G(S)$ is the group of units of S;

(iii) the congruence $\sim$ identifies elements of $J(A)\cup G(S)$ which act identically on $ST(A)$;

(iv) $X(A)$ is regular and satisfies ACC on cyclic left ideals.

Proof. Let $n=d(T)$ be the depth of the transformation semigroup T. We prove by transfinite induction that for every $1\leqq j\leqq n$ there is a relational covering $T\leqq_\alpha \ldots \circ \overline{\mathbf{T}}_1^{\vee}(T)\circ \overline{\mathbf{T}}_0^{\vee}(T)$ of depth j.

Base case $j=1$. This is theorem 5.6.

Inductive step. Assume that there is an ordinal J such that the theorem is true for all $j<J$, and assume that $j=J$. There are two cases to consider.

Case I: J is a non-limit ordinal. Then the result follows from theorem 5.6.

Case II: J is a limit ordinal. Note that $\pi^J=\bigcap_{k<J}\pi^k$. Therefore, the image of $\ldots\circ\overline{\mathbf{T}}_1^{\vee}(T)\circ\overline{\mathbf{T}}_0^{\vee}(T)$ is π^J, where the terms in this product are indexed by all the ordinals j such that $0\leqq j<J$. This proves the existence of the relational covering.

Now each $\overline{\mathbf{T}}_j^{\vee}(T)=T_j^{\vee}(T)\vee(K, C)$ where $K=\bigcup_{k\geqq 1}A_k^j$ and C is the set of constant maps on K. We can in turn decompose (K, C) as $(A_1^j, C_1^j)\vee\ldots\vee(A_k^j, C_k^j)\vee\vee\ldots$ and $T_j^{\vee}(T)$ as $T(A_1^j)\vee\ldots\vee T(A_k^j)\ldots$. The remainder of the theorem follows from proposition 4.5. ∎

If we assume that S has a composition series — that is, a sequence of two sided ideals $I_1\supset I_2\supset\ldots\supset I_n\supset\ldots$ such that each I_{j+1} is a maximal ideal in the semigroup I_j — then it seems possible to replace the factors $T(A_k^j)$ of the decomposition by simple semigroups. (The author has not checked this fact. For a possible method of proof of this conjecture, see Tucci [17].)

We conclude with two trivial examples to show that the decomposition can be either finite or infinite.

5.10. Example. Consider the transformation semigroup of example 2.2. Any skeleton element is either equivalent to Q or is a singleton. Hence the depth of the transformation semigroup T is 1, and so $T\leqq(Q, \overline{\mathbf{Z}})=\overline{\mathbf{T}}$; that is, the decomposition is trivial in this case. ∎

5.11. Example. Let $T=(Q, S)$ where $Q=\{q_n|n\in N\}$, and

$$S=\langle(x_j, x_j^{-1}, a_j, a_j^{-1}|j\in N)\rangle$$

where

(1) $F_{x_n}(q_n)=q_{n+1}$ for all $n\geqq 1$;

(2) $F_{a_k}(q_n)=q_n$ for all $1\leqq k\leqq n$.

The functions induced by all x_j^{-1} and a_j^{-1}, $j\geqq 1$, are defined in the obvious manner. For all other $s\in S$ and $q\in Q$, the expression $F_s(q)$ is undefined.

The skeleton elements of T are either singletons or of the form $A_n=\{q_j|j\geqq n$ for some integer $n\}$. Hence there is an infinite descending chain of non-equivalent skeleton elements $Q=A_1\supset A_2\supset\ldots$ which yields an infinite decomposition $T\leqq_\alpha\ldots$ $\ldots\circ\bar{\mathbf{T}}_1^{\vee}(T)\circ\bar{\mathbf{T}}_0^{\vee}(T)$. ∎

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