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Modelling of a Communication System Evolving in a Random Environment

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Abstract

This paper is concerned with a queueing model to analyse the asymptotic behaviour of a finite-source communication system with a receiver containing multiple processors of the same kind. The source and processing times of each message are supposed to be exponentially distributed random variables with parameter depending on the state of a varying environment. Assuming that the arrivals of the messages are "fast" compared to their service, it is shown that the time to the first system failure converges in distribution, under appropriate norming, to an exponentially distributed random variable.

Keywords: queueing, communication system, reliability, weak convergence.

1 Introduction

Performance evaluation of information system development has become more complex as the size and complexity of the system has increased, see Takagi (1990). Reliability is certainly the most important characteristic for communication networks. The measure of greatest interest is the distribution of the time to the first system failure. It is well-known that the majority of the problems can be treated by the help of Semi-Markov Processes (SMP) or Semi-Regenerative Processes (SRP). Since the failure-free operation time of the system correponds to sojourn time problems we can use the results obtained for SMP, cf. Ushakov (1985), Osaki et al. (1987). If the exit from a given subset of the state space is a "rare" event, that is, it occurs with a small probability it is natural to investigate the asymptotic behaviour of the sojourn time in that subspace, see Gertsbakh (1984, 1989), Keilson (1979), Rukhin and Hsieh (1987).

This paper is concerned with a queueing model to analyse the asymptotic behaviour of a finite-source communication system with a receiver containing multiple processors of the same kind. The source and processing times of each message

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are supposed to be exponentially distributed random variables with parameter depending on the state of a varying environment. Assuming that the arrivals of the messages are "fast" compared to their service, it is shown that the time to the first system failure converges in distribution, under appropriate norming, to an exponentially distributed random variable.

2 Preliminary results

In this section a brief survey is given of the most related theoretical results, mainly due to Anisimov, to be applied later on.

Let $(X_\varepsilon(k), k \geq 0)$ be a Markov chain with state space

$$\bigcup_{q=0}^{m+1} X_q, \quad X_i \cap X_j = 0, \quad i \neq j,$$

defined by the transition matrix $\|p_\varepsilon(i^{(q)}, j^{(z)})\|$ satisfying the following conditions:

1. $p_\varepsilon(i^{(0)}, j^{(0)}) \longrightarrow p_0(i^{(0)}, j^{(0)}), as \quad \varepsilon \longrightarrow 0,$

   $i^{(0)}, j^{(0)} \in X_0$, and $P_0 = \|p_0(i^{(0)}, j^{(0)})\|$ is irreducible;

2. $p_\varepsilon(i^{(q)}, j^{(q+1)}) = \varepsilon\alpha^{(q)}(i^{(q)}, j^{(q+1)}) + o(\varepsilon), \quad i^{(q)} \in X_q, \quad j^{(q+1)} \in X_{q+1};$

3. $p_\varepsilon(i^{(q)}, f^{(q)}) \longrightarrow 0$, as $\varepsilon \longrightarrow 0, \quad i^{(q)}, f^{(q)} \in X_q, q \geq 1;$

4. $p_\varepsilon(i^{(q)}, f^{(z)}), \equiv 0, \quad i^{(q)} \in X_q, f^{(z)} \in X_z, \quad z - q \geq 2.$

In the sequel the set of states X_q is called the q-th level of the chain, $q = 1, \ldots, m+1$. Let us single out the subset of states

$$\langle \alpha_m \rangle = \bigcup_{q=0}^{m} X_q.$$

Denote by $\{\pi_\varepsilon(i^{(q)}), i^{(q)} \in X_q\}, q = 1, \ldots, m$ the stationary distribution of a chain with transition matrix

$$\left\| \frac{p_\varepsilon(i^{(q)}, j^{(z)})}{1 - \sum\limits_{k^{(m+1)} \in X_{m+1}} p_\varepsilon(i^{(q)}, k^{(m+1)})} \right\|, i^{(q)} \in X_q, j^{(z)} \in X_z, q, z \leq m,$$

furthermore denote by $g_\varepsilon(\langle \alpha_m \rangle)$ the steady state probability of exit from $\langle \alpha_m \rangle$, that is

$$g_\varepsilon(\langle \alpha_m \rangle) = \sum_{i^{(m)} \in X_m} \pi_\varepsilon(i^{(m)}) \sum_{j^{(m+1)} \in X_{m+1}} p_\varepsilon(i^{(m)}, j^{(m+1)}).$$

Denote by $\{\pi_0(i^{(0)}), i^{(0)} \in X_0\}$ the stationary distribution corresponding to P_0 and let

$$\overline{\pi}_0 = \{\pi_0(i^{(0)}), i^{(0)} \in X_0\}, \quad \overline{\pi}_\varepsilon^{(q)} = \{\pi_\varepsilon(i^{(q)}), i^{(q)} \in X_q\}$$

be row vectors. Finally, let

$$A^{(q)} = \|\alpha^{(q)}(i^{(q)}, j^{(q+1)})\|, \quad i^{(q)} \in X_q, j^{(q+1)} \in X_{q+1}, q = 0, \ldots, m$$

defined by Condition 2.

Conditions (1)-(4) enables us to compute the main terms of the asymptotic expression for $\pi_\varepsilon^{(q)}$ and $g_\varepsilon(\langle\alpha_m\rangle)$. Namely, we obtain

$$\overline{\pi}_\varepsilon^{(q)} = \varepsilon^q \overline{\pi}_0 A^{(0)} A^{(1)} \ldots A^{(q-1)} + o(\varepsilon^q) \quad q = 1, \ldots, m,$$

$$g_\varepsilon(\langle\alpha_m\rangle) = \varepsilon^{m+1} \overline{\pi}_0 A^{(0)} A^{(1)} \ldots A^{(m)} \underline{1} + o(\varepsilon^{m+1}), \tag{1}$$

where $\underline{1} = (1, \ldots, 1)^*$ is a column vector, see Anisimov et al. (1987) pp. 141-153. Let $(\eta_\varepsilon(t), \quad t \geq 0)$ be a SMP given by the embedded Markov chain $(X_\varepsilon(k) \quad k \geq 0)$ satisfying conditions (1)-(4). Let the times $\tau_\varepsilon(j^{(s)}, k^{(z)})$– transition times from state $j^{(s)}$ to state $k^{(z)}$– fulfil the condition

$$\mathbf{E}\exp\{i\Theta\beta_\varepsilon\tau_\varepsilon(j^{(s)}, k^{(z)})\} = 1 + a_{jk}(s, z, \Theta)\varepsilon^{m+1} + o(\varepsilon^{m+1}), \quad (i^2 = -1)$$

where β_ε is some normalizing factor. Denote by $\Omega_\varepsilon(m)$ the instant at which the SMP reaches the $m+1$-th level for the first time, exit time from $\langle\alpha_m\rangle$, provided $\eta_\varepsilon(0) \in \langle\alpha_m\rangle$. Then we have:

Theorem 1 *(cf. Anisimov et al. (1987) pp. 153) If the above conditions are satisfied then*

$$\lim_{\varepsilon \longrightarrow 0} \mathbf{E}\exp\{i\Theta\beta_\varepsilon\Omega_\varepsilon(m)\} = (1 - A(\Theta))^{-1},$$

where

$$A(\Theta) = \frac{\sum_{j^{(0)}, k^{(0)} \in X_0} \pi_0(j^{(0)}) p_0(j^{(0)}, k^{(0)}) a_{jk}(0, 0, \Theta)}{\overline{\pi}_0 A^{(0)} A^{(1)} \ldots A^{(m)} \underline{1}}.$$

Corollary 1 *In particular, if $a_{jk}(s, z, \Theta) = i\Theta m_{jk}(s, z)$ then the limit is an exponentially distributed random variable with mean*

$$\frac{\sum_{j^{(0)}, k^{(0)} \in X_0} \pi_0(j^{(0)}) p_0(j^{(0)}, k^{(0)}) m_{jk}(0, 0)}{\overline{\pi}_0 A^{(0)} A^{(1)} \ldots A^{(m)} \underline{1}}.$$

3 The mathematical model

Let us consider a communication system consisting of N sources of information and n processors of the same kind at the receiver. The whole system is assumed to operate in a random environment governed by an ergodic Markov chain $(\xi(t), t \geq 0)$ with state space $(1, \ldots, r)$ and with transition density matrix $(a_{ij}, i, j = 1, \ldots, r, a_{ii} = \sum_{j \neq i} a_{ij})$. Whenever the environmental process is in state i the probability that an active source generates a message in the time interval $(t, t+h)$ is $\lambda(i, \varepsilon)h + o(h)$. Each message is transmitted to a receiver where the service immediately starts if there is an idle processor, otherwise a queueing line is formed. The service discipline is First Come-First Served (FCFS). Whenever the environmental process is in state i the probability that the processing of a given message is completed in time interval $(t, t+h)$ is $\mu(i)h + o(h)$. If a given source has sent a message it stays idle and it cannot generate other one. After being serviced each message immediately returns to its source which hence becomes active. All random variables involved her and the random environment are supposed to be independent of each other.

In practical applications it is very important to know the distribution of time until the receiver becomes empty.

Let us consider the system unnder the assumption of "fast" arrivals, i.e., $\lambda(i,\varepsilon) \longrightarrow \infty$ as $\varepsilon \longrightarrow 0$. For simplicity let $\lambda(i,\varepsilon) = \lambda(i)/\varepsilon$. Denote by $Y_\varepsilon(t)$ the number of active sources at time t, and let

$$\Omega_\varepsilon(m) = \inf\{t : t > 0, Y_\varepsilon(t) = m + 1 | Y_\varepsilon(0) \leq m\},$$

that is, the instant at which the number of active sources reaches the $m+1$-th level for the first time, provided that at the begining their number is not greater than m, $m = 0, \ldots, N-1$. In the following $\Omega_\varepsilon(m)$ is referred to as the time to the first system failure. In particular, if $m = N-1$ than the receiver becomes empty.

Denote by $(\pi_k, k = 1, \ldots, r)$ the steady-state distribution of the governing Markov chain $(\xi(t), t \geq 0)$. Now we have:

Theorem 2 *For the system in question under the above assumptions, independently of the initial state, the distribution of the normalized random variable $\varepsilon^m \Omega_\varepsilon(m)$ converges weakly to an exponentially distributed random variable with parameter*

$$\Lambda = \frac{n}{m!} \prod_{s=1}^{m} \min(n, N-s) \sum_{i=1}^{r} \pi_i \frac{\mu(i)^{m+1}}{\lambda(i)^m}.$$

Proof. It is easy to see that the process

$$Z_\varepsilon(t) = (\xi(t), Y_\varepsilon(t))$$

is a two-dimensional Markov chain with state space

$$E = ((i,x), \quad i = 1, \ldots, r, \quad s = 0, \ldots N).$$

Furthermore, let

$$\langle \alpha_m \rangle = ((i,s), \quad i = 1, \ldots, r, \quad s = 0, \ldots, m).$$

Hence our aim is to determine the distribution of the first exit time of $Z_\varepsilon(t)$ from $\langle \alpha_m \rangle$, provided that $Z_\varepsilon(\text{o}) \in \langle \alpha_m \rangle$.

It can easily be verified that the transition probabilities in any time interval $(t, t+h)$ are the following:

$$(i,s) \xrightarrow{h} \begin{cases} (j,s) & a_{ij}h + o(h), & i \neq j, \\ (i,s+1) & \min(n, N-s)\mu(i)h + o(h) & s = 0, \ldots, N-1, \\ (i,s-1) & (s\lambda(i)/\varepsilon)h + o(h), & s = 1, \ldots, N. \end{cases}$$

In addition, the sojourn time $\tau_\varepsilon(i,s)$ of $Z_\varepsilon(t)$ in state (i,s) is exponentially distributed with parameter $a_{ii} + s\lambda(i)/\varepsilon + \min(N, n-s)\mu(i)$. Thus, the transition probabilities for the embedded Markov chain are

$$p_\varepsilon[(i,s),(j,s)] = \frac{a_{ij}}{a_{ii} + s\lambda(i)/\varepsilon + \min(n, N-s)\mu(i)}, \quad s = 0,\dots,N,$$

$$p_\varepsilon[(i,s),(i,s+1)] = \frac{\min(n, N-s)\mu(i)}{a_{ii} + s\lambda(i)/\varepsilon + \min(n, N-s)\mu(i)}, \quad s = 0,\dots,N-1,$$

$$p_\varepsilon[(i,s),(i,s-1)] = \frac{s\lambda(i)/\varepsilon}{a_{ii} + s\lambda(i)/\varepsilon + \min(n, N-s)\mu(i)}, \quad s = 1,\dots,N.$$

As $\varepsilon \longrightarrow 0$ this implies

$p_\varepsilon[i,0),(j,0)] = \frac{a_{ij}}{a_{ii}+n\mu(i)}, \quad s = 0,$

$p_\varepsilon[(i,s),(j,s)] = o(1), \quad s = 1,\dots,N,$

$p_\varepsilon[(i,0),(i,1)] = \frac{n\mu(i)}{a_{ii}+n\mu(i)}, \quad s = 0,$

$p_\varepsilon[(i,s),(i,s+1)] = \frac{\min(n,N-s)\mu(i)\varepsilon}{s\lambda(i)}(1 + o(\varepsilon)), \quad s = 1,\dots,N-1,$

$p_\varepsilon[(i,s),(i,s-1)] \longrightarrow 1, \quad s = 1,\dots N.$

This agrees with the conditions (1)-(4), but here the zero level is the set $((i,0),(i,1), i = 1,\dots,r)$ while the q-th level is $((i,q+1), i = 1,\dots,r)$. Since the level 0 in the limit forms an essential class, the probabilities $\pi_0(i,0), \pi_0(i,1)$, $i = 1,\dots,r$ satisfy the following system of equations

$$\pi_0(j,0) = \sum_{i\neq j} \pi_0(i,0)a_{ij}/(a_{ii} + n\mu(i)) + \pi_0(j,1) \tag{2}$$

$$\pi_0(j,1) = \pi_0(j,0)n\mu(j)/(a_{jj} + n\mu(j)). \tag{3}$$

By substituting (3) to (2) we get

$$\pi_0(j,0)a_{jj}/(a_{jj} + n\mu(j)) = \sum_{i\neq j} \pi_0(i,0)a_{ij} \setminus (a_{ii} + n\mu(i)). \tag{4}$$

Since

$$\pi_j a_{jj} = \sum_{i\neq j} \pi_i a_{ij},$$

from (3) and (4) we have

$$\pi_0(i,0) = B\pi_i(a_{ii} + n\mu(i)), \quad \pi_0(i,1) = B\pi_i n\mu(i),$$

where B is the normalizing constant, i.e.

$$1/B = \sum_{i=1}^{r} \pi_i [a_{ii} + 2n\mu(i)].$$

By using (1) it is easy to show that the probability of exit from $\langle \alpha_m \rangle$ is

$$\begin{aligned} g_\varepsilon(\langle \alpha_m \rangle) &= \varepsilon^m n B \sum_{i=1}^{r} \pi_i \mu(i) \prod_{s=1}^{m} \frac{\min(n, N-s)\mu(i)}{s\lambda(i)} (1 + o(1)) \\ &= \frac{\varepsilon^m n B}{m!} \prod_{s=1}^{m} \min(n, N-s) \sum_{i=1}^{r} \pi_i \frac{\mu(i)^{m+1}}{\lambda(i)^m} (1 + o(1)). \end{aligned}$$

Taking into account the exponentiality of $\tau_\varepsilon(j, s)$ for fixed Θ we have

$$\mathbf{E} \exp \{i\varepsilon^m \Theta \tau_\varepsilon(j, 0)\} = 1 + \varepsilon^m \frac{i\Theta}{a_{jj} + n\mu(j)} (1 + o(1))$$

$$\mathbf{E} \exp \{i\varepsilon^m \Theta \tau_\varepsilon(j, s)\} = 1 + o(\varepsilon^m), \quad s > 0.$$

Notice that $\beta_\varepsilon = \varepsilon^m$ and therefore from Corollary 1 we immediately get the statement that $\varepsilon^m \Omega_\varepsilon(m)$ converges weakly to an exponentially distributed random variable with parameter

$$\Lambda = \frac{n}{m!} \prod_{s=1}^{m} \min(n, N-s) \sum_{i=1}^{r} \pi_i \frac{\mu(i)^{m+1}}{\lambda(i)^m},$$

which completes the proof.

Consequently, the distribution of the time to the first system failure can be approximated by

$$P(\Omega_\varepsilon(m) > t) = P(\varepsilon^m \Omega_\varepsilon(m) > \varepsilon^m t) \approx \exp(-\varepsilon^m \Lambda t),$$

i.e. $\Omega_\varepsilon(m)$ is asymptotically an exponenetially distributed random variable with parameter $\varepsilon^m \Lambda$. In particular, for $m = N - 1$ we have

$$\varepsilon^{N-1} \Lambda = \frac{\varepsilon^{N-1} n!}{(N-1)!} n^{N-n} \sum_{i=1}^{r} \pi_i \frac{\mu(i)^N}{\lambda(i)^{N-1}} = \frac{n!}{(N-1)!} n^{N-n} \sum_{i=1}^{r} \pi_i \frac{\mu(i)^N}{(\lambda(i)/\varepsilon)^{N-1}}.$$

In the case when there is no random environment we get

$$\varepsilon^{N-1} \Lambda = \frac{n!}{(N-1)!} n^{N-n} \frac{\mu^N}{(\lambda/\varepsilon)^{N-1}}.$$

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