Acta Cybernetica 13 (1998) 319-328.

On Decision-Mappings Related to Process Network Synthesis Problem

Z. Blázsik * Cs. Holló† B. Imreh†

Abstract

Process network synthesis (PNS) has enormous practical impact and a structural model can be given for it on the basis of a combinatorial approach. An important tool of this approach is the notion of the decision-mapping. In the present work, the number of the consistent decision-mappings is counted and an upper bound is presented for the number of the feasible solutions of a PNS problem.

Introduction

In a manufacturing system, materials of different properties are converted into desired products through various physical, chemical, and biological transformations. Devices in which these transformations are carried out are called operating units and a manufacturing system can be considered as a network of operating units, i.e., *process network*. Naturally, minimizing the cost of a process network is indeed essential. For this purpose, several papers have appeared for solving PNS problems by global optimization methods (see, e.g., [2] and [8]) and by combinatorial approach based on the feasible graphs of processes (cf. [3], [4], [5], [7], and [9]).

In this paper, using the combinatorial approach, the number of the consistent decision-mappings is counted, furthermore, an upper bound is given for the number of the feasible solutions of a PNS problem. The paper is organized as follows: Section 1 reviews the precise definition of the structural model of PNS problem and introduces some relevant basic concepts. In Section 2, the number of the consistent decision-mappings over a nonempty set is calculated. On the basis of the relationship between the maximal consistent decision-mappings and the feasible solutions, an upper bound is presented for the number of the feasible solutions in Section 3. Finally, Section 4 contains an example for illustrating this bound.

*Research Group on Artificial Intelligence of the Hungarian Academy of Sciences, Aradi vértanúk tere 1, H-6720 Szeged, Hungary

†Dept. of Informatics, József Attila University, Árpád tér 2, H-6720 Szeged, Hungary

1 Preliminaries

Let M be a given finite set of objects which are materials capable of being converted or transformed by a process. Transformation between two subsets of M occurs in an operating unit. It is necessary to link this operating unit to others through the elements of these two subsets of M. The resultant structure is a process graph (see [4] and [5]) defined as follows.

Let M be a finite nonempty set, and also let $O \subseteq \wp'(M) \times \wp'(M)$ with $O \neq \emptyset$ and $M \cap O = \emptyset$, where $\wp'(M)$ denotes the set of all nonempty subsets of M. The elements of O are called *operating units*: for operating unit $u = (\alpha, \beta) \in O$, α and β are called the *input-set* and *output-set* of u, respectively. Pair (M, O) is defined as *process graph* or P-*graph* in short. The set of vertices of this directed graph is $M \cup O$, and the set of arcs is $A = A_1 \cup A_2$ with $A_1 = \{(X, Y) : Y = (\alpha, \beta) \in O \text{ and } X \in \alpha\}$ and $A_2 = \{(Y, X) : Y = (\alpha, \beta) \in O \text{ and } X \in \beta\}$. If there exist vertices $X_1, X_2, ..., X_n$, such that $(X_1, X_2), (X_2, X_3), \ldots, (X_{n-1}, X_n)$ are arcs of process graph (M, O), then $[X_1, X_n]$ is defined to be a *path* from vertex X_1 to vertex X_n. Let process graphs (m, o) and (M, O) be given; (m, o) is defined to be a *subgraph* of (M, O), if $m \subseteq M$ and $o \subseteq O$.

To define a structural model of PNS, the set of materials to be included in the model need be specified. In the sequal, each material is an element of M^*, an arbitrarily specified infinite set of the available materials. From the technical point of view, we suppose that $M^* \cap (\wp'(M^*) \times \wp'(M^*)) = \emptyset$. Now, a process design problem can be defined from a structural point of view in the following way. By a *structural model* of PNS, we mean the triplet, $\mathbf{M} = (P, R, O)$, where $P \subseteq M^*$ and $O \subseteq \wp'(M^*) \times \wp'(M^*)$ are finite nonempty sets representing the set of desired products and that of available operating units, respectively, and $R \subseteq M^*$ is a finite set representing the set of raw materials. Moreover, $P \cap R = \emptyset$, and α, β are finite sets for all operating units $u = (\alpha, \beta) \in O$.

Now, let $\mathbf{M} = (P, R, O)$ be a structural model of PNS; then, we can assign a P-graph to $\mathbf{M}$ as follows. Let M' denote the set of materials belonging to the operating units from O and M denote set $M' \cup P \cup R$. It can be seen that M and O are nonempty finite sets and that $O \subseteq \wp'(M) \times \wp'(M)$ and $M \cap O = \emptyset$. Thus, (M, O) is a P-graph representing the interconnections among the operating units in set O. Since $M \cap O = \emptyset$, the vertices which are the points in (M, O) can be divided into the two disjoint sets, M and O. The elements of M are called *material points* and those of O, *unit points* of (M, O). A subgraph of (M, O) can be assigned to each feasible process of the PNS problem; such a subgraph represents the structure or network of the process under consideration. If additional constraints, e.g., the material balance, are disregarded, the subgraphs of (M, O), which can be assigned to the feasible processes, have common combinatorial properties. Such properties, explored in [5], are given below.

Subgraph (m, o) of (M, O) is called a *feasible solution* of $\mathbf{M} = (P, R, O)$ if the following properties are satisfied:

(A1) $P \subseteq m$,

(A2) $\forall X \in m$, $X \in R \Leftrightarrow$ there exists no (Y, X) arc in (m, o),

(A3) $\forall Y_0 \in o$, $\exists$ path $[Y_o, Y_n]$ with $Y_n \in P$,

(A4) $\forall X \in m$, $\exists(\alpha, \beta) \in o$ such that $X \in \alpha \cup \beta$.

Let us denote the set of the feasible solutions of $\mathbf{M}$ by $S(\mathbf{M})$. It is easy to see that $S(\mathbf{M})$ is closed under the finite union. Consequently,

$$\cup\{(m, o) : (m, o) \in S(\mathbf{M})\}$$

is also a feasible solution provided that $S(\mathbf{M}) \neq \emptyset$; it is the greatest feasible solution with respect to the relation, subgraph ordering. This distinguished graph is called the *maximal structure* of $\mathbf{M}$.

Now, a simple class of PNS problems can be defined, a class of such PNS problems in which each operating unit has a positive fixed charge. We are to find a feasible process with the minimum cost; by the cost of a process, we mean the sum of the fixed charges of the operating units belonging to the process of interest. Each feasible process in this class of PNS problems is uniquely determined from the corresponding feasible solution and vice versa. Hence, the problem under consideration can be formalized as follows.

Let a structural model of PNS problem $\mathbf{M} = (P, R, O)$ be given; moreover, let z be a positive real-valued function defined on O. The basic model is then

$$\text{(i)} \qquad \min\{\sum_{u \in o} z(u) : (m, o) \in S(\mathbf{M})\}.$$

It has been proved [1] that this PNS problem is NP-hard; therefore, the branch-and-bound technique may be a possible tool for its solution (see [7] and [9]).

2 Consistent decision-mappings

In the branch-and-bound procedures for solving PNS problems, the notion of the decision-mapping (see [6]) has been applied. Let $\mathbf{M} = (P, R, O)$ be a structural model of PNS. Then, P-graph (M, O) of $\mathbf{M}$ determines a function Δ of $M \setminus R$ into $\wp'(O)$ as follows. For any material $X \in M \setminus R$, let

$$\Delta(X) = \{(\alpha, \beta) : (\alpha, \beta) \in O \;\; \& \;\; X \in \beta\}.$$

Let m be a subset of $M \setminus R$; furthermore, let $\delta(X)$ be a subset of $\Delta(X)$ for each $X \in m$. Mapping δ from set m into the set of subsets of O, $\delta[m] = \{(X, \delta(X)) : X \in m\}$, is called a *decision-mapping belonging* to $\mathbf{M}$; $\delta[m]$ is said to be *consistent* when $\delta(X) \cap \Delta(Y) \subseteq \delta(Y)$ is valid for all $X, Y \in m$, and the set of all consistent decision-mappings of $\mathbf{M}$ is denoted by $\Omega_{\mathbf{M}}$. In particular, if $\delta[m] \in \Omega_{\mathbf{M}}$ and $m = M \setminus R$, then sometimes we use the shorter notation δ instead of $\delta[M \setminus R]$.

A decision-mapping can be visualised as a sequence of decisions, each of which is concerned with a single material involved in the process being synthesized; it identifies the set of operating units to be considered for producing directly the material of interest. The meaning of the consistency can be presented as follows. Material X is to be produced by operating units included in $\delta(X)$. Then, those operating units of $\delta(X)$ that also participate in the production of material Y, i.e., $\delta(X) \cap \Delta(Y)$, must be considered for the production of material Y, and thus, $\delta(Y) \supseteq \delta(X) \cap \Delta(Y)$.

We define function *op* on $\Omega_{\mathbf{M}}$ for selecting the set of those operating units that are decided to produce any of the materials in set m based on consistent decision-mapping $\delta[m]$. Formally, for any $\delta[m] \in \Omega_{\mathbf{M}}$,

$$op(\delta[m]) = \cup\{\delta(X) : X \in m\}.$$

In what follows, we need the following functions. For any finite set of operating units o, let

$$mat^{in}(o) = \cup\{\alpha : (\alpha, \beta) \in o\}, \qquad mat^{out}(o) = \cup\{\beta : (\alpha, \beta) \in o\}.$$

Let $\delta_1[m_1]$ and $\delta_2[m_2]$ be arbitrary consistent decision-mappings. Then, $\delta_2[m_2]$ is called an *extension* of $\delta_1[m_1]$ if $m_1 \subseteq m_2$ and $\delta_1(X) = \delta_2(X)$ for all $X \in m_1$; this is denoted by $\delta_1[m_1] \leq \delta_2[m_2]$. Relation extension is reflexive, antisymmetric and transitive; hence, it is a partial ordering on $\Omega_{\mathbf{M}}$. Let us denote the set of all maximal elements of this partially ordered set by $\Omega'_{\mathbf{M}}$. Regarding the number of the consistent decision-mappings over a nonempty set m, the following statement is valid.

Theorem. *For every $\emptyset \neq m \subseteq M \setminus R$, the number of the consistent decision-mappings defined on m is $2^{|\cup\{\Delta(X):X\in m\}|}$.*

Proof. We proceed by induction on $|m|$. If $|m| = 1$, then $X \to Q$ is a consistent decision-mapping for every subset Q of $\Delta(X)$ where X denotes the single element of m. Therefore, the required number is $2^{|\Delta(x)|}$.

Now, let $1 \leq i < |M \setminus R|$ be an arbitrary integer, and let us suppose that the statement is valid for all $m \subseteq M \setminus R$ with $|m| = i$. Let us consider an arbitrary subset m' ($\subseteq M \setminus R$) consisting of $i+1$ elements. Without loss of generality, it can be assumed that $m' = \{X_1, \ldots, X_i, X_{i+1}\}$. Let $W = \Delta(X_{i+1}) \setminus (\cup\{\Delta(X_t) : t = 1, \ldots, i\})$. The following two cases are distinguished depending on W.

Case 1. $W = \emptyset$. From the definition of the consistent decision-mapping, the following observation can be directly obtained. For each consistent decision-mapping $\delta[m']$, the restriction of $\delta[m']$ to set $\{X_1, \ldots, X_i\}$ is also consistent decision-mapping. On the other hand, if two consistent decision-mappings defined on the same set are different, then their extensions constitute two disjoint sets. In the light of these observations, it is enough to prove that consistent decision-mapping $\delta[\{X_1, \ldots, X_i\}]$ has one and only one extension to $\{X_1, \ldots, X_i, X_{i+1}\}$.

First, we construct an extension of $\delta[\{X_1, \ldots, X_i\}]$ to $\{X_1, \ldots, X_i, X_{i+1}\}$. The new decision-mapping is defined as follows. Let

$\delta'(X_{i+1}) = \{(\alpha,\beta) : (\alpha,\beta) \in \Delta(X_{i+1})\ \&\ (\alpha,\beta) \in \delta(X_j) \text{ for some } j \in \{1,\dots,i\}\}$,
and

$\delta'(X_t) = \delta(X_t)$ for all t, $t = 1,\dots,i$.

Regarding the consistency of $\delta'[\{X_1,\dots,X_i,X_{i+1}\}]$, we have to prove that

(1) $\quad \delta'(X_t) \cap \Delta(X_{i+1}) \subseteq \delta'(X_{i+1})$,

and

(2) $\quad \delta'(X_{i+1}) \cap \Delta(X_t) \subseteq \delta'(X_t)$

are valid for all $X_t \in \{X_1,\dots,X_i\}$. The validity of (1) follows from the definition of δ'. For verifying (2), let $(\alpha,\beta) \in \delta'(X_{i+1}) \cap \Delta(X_t)$ for some $t \in \{1,\dots,i\}$. Since $(\alpha,\beta) \in \delta'(X_{i+1})$, there exists a $j \in \{1,\dots,i\}$ with $(\alpha,\beta) \in \delta(X_j) \cap \Delta(X_{i+1})$. Then, $(\alpha,\beta) \in \delta(X_j) \cap \Delta(X_t)$. On the other hand, $j,t \in \{1,\dots,i\}$ and the consistency of δ results in $\delta(X_j) \cap \Delta(X_t) \subseteq \delta(X_t) = \delta'(X_t)$. Consequently, $(\alpha,\beta) \in \delta'(X_t)$ yielding the validity of (2).

Now, let us suppose that decision-mapping $\delta^*[\{X_1,\dots,X_i,X_{i+1}\}]$ is an extension of $\delta[\{X_1,\dots,X_i\}]$. We show that $\delta'(X_t) = \delta^*(X_t)$ is valid for all t, $t = 1,\dots,i+1$. If $1 \le t \le i$, then the required equality obviously holds. Therefore, it is enough to prove that $\delta'(X_{i+1}) \subseteq \delta^*(X_{i+1})$ and $\delta'(X_{i+1}) \supseteq \delta^*(X_{i+1})$. To do so, let $(\alpha,\beta) \in \delta'(X_{i+1})$ be an arbitrary operating unit. By the definition of δ', $(\alpha,\beta) \in \delta(X_j) \cap \Delta(X_{i+1})$ for some $X_j \in \{X_1,\dots,X_i\}$. But $\delta(X_j) = \delta^*(X_j)$ and δ^* is consistent. Consequently,

$$(\alpha,\beta) \in \delta^*(X_j) \cap \Delta(X_{i+1}) \subseteq \delta^*(X_{i+1}).$$

Conversely, let $(\alpha,\beta) \in \delta^*(X_{i+1})$. Since $W = \emptyset$, there exists a $j \in \{1,\dots,i\}$ such that $(\alpha,\beta) \in \Delta(X_j)$, and thus, $(\alpha,\beta) \in \delta^*(X_{i+1}) \cap \Delta(X_j)$. Now, by the consistency of δ^*, $\delta^*(X_{i+1}) \cap \Delta(X_j) \subseteq \delta^*(X_j) = \delta(X_j)$. Therefore, $(\alpha,\beta) \in \Delta(X_{i+1}) \cap \delta(X_j)$, but then, $(\alpha,\beta) \in \delta'(X_{i+1})$ from the definition of δ'.

Case 2. $W \neq \emptyset$. Using the observations of Case 1 again, it is sufficient to prove that consistent decision-mapping $\delta[\{X_1,\dots,X_i\}]$ has $2^{|W|}$ extensions to $\{X_1,\dots,X_i,X_{i+1}\}$. For this purpose, let

$$T = \{(\alpha,\beta) : (\alpha,\beta) \in \Delta(X_{i+1})\ \&\ (\alpha,\beta) \in \delta(X_t) \text{ for some } t \in \{1,\dots,i\}\}.$$

From the definitions, $T \cap W = \emptyset$. Now, we show that decision-mapping δ' defined by

$$\delta'(X) = \begin{cases} \delta(X) & \text{if } X \in \{X_1,\dots,X_i\}, \\ T \cup Q & \text{if } X = X_{i+1}, \end{cases}$$

is consistent for every subset Q of W. Since $\delta[\{X_1,\dots,X_i\}]$ is consistent, we have to prove that the following inclusions

(3) $\quad \delta'(X_j) \cap \Delta(X_{i+1}) \subseteq \delta'(X_{i+1})$,

and

(4) $\quad \delta'(X_{i+1}) \cap \Delta(X_j) \subseteq \delta'(X_j)$,

are valid for all j, $j = 1, \ldots, i$.

To prove these inclusions, let $j \in \{1, \ldots, i\}$ be an arbitrary index. First, let $(\alpha, \beta) \in \delta'(X_j) \cap \Delta(X_{i+1})$. Then, $(\alpha, \beta) \in T$, and thus, $(\alpha, \beta) \in \delta'(X_{i+1})$ resulting in (1). Now, let $(\alpha, \beta) \in \delta'(X_{i+1}) \cap \Delta(X_j)$. Then, $(\alpha, \beta) \in (T \cup Q) \cap \Delta(X_j) = T \cap \Delta(X_j)$. Inclusion $(\alpha, \beta) \in T$ implies that $(\alpha, \beta) \in \delta(X_t)$ for some $t \in \{1, \ldots, i\}$. Consequently, $(\alpha, \beta) \in \delta(X_t) \cap \Delta(X_j)$. Since δ is consistent, $\delta(X_t) \cap \Delta(X_j) \subseteq \delta(X_j) = \delta'(X_j)$ which yields (4).

By the construction above, $2^{|W|}$ different extensions of $\delta[\{X_1, \ldots, X_i\}]$ are presented. To complete the proof, it is shown that the decision-mapping under consideration has no further extensions to $\{X_1, \ldots, X_i, X_{i+1}\}$. Indeed, let $\delta'[\{X_1, \ldots, X_i, X_{i+1}\}]$ be an arbitrary extension of δ and $(\alpha, \beta) \in T$. Then, $(\alpha, \beta) \in \delta(X_t) \cap \Delta(X_{i+1}) = \delta'(X_t) \cap \Delta(X_{i+1})$ for some $t \in \{1, \ldots, i\}$. Since δ' is consistent, $\delta'(X_t) \cap \Delta(X_{i+1}) \subseteq \delta'(X_{i+1})$, and thus, $(\alpha, \beta) \in \delta'(X_{i+1})$. Consequently, $T \subseteq \delta'(X_{i+1})$. On the other hand, $(\alpha, \beta) \in \delta'(X_{i+1}) \setminus T$ implies $(\alpha, \beta) \in W$. In the opposite case, $(\alpha, \beta) \in \Delta(X_t)$ for some $t \in \{1, \ldots, i\}$, and then, $(\alpha, \beta) \in \delta'(X_t) = \delta(X_t)$ because of the consistency of δ' which is a contradiction. Then, $\delta'(X_{i+1}) \subseteq T \cup W$, and thus, δ' is equal to one of the given extensions of δ.

Now, by the induction hypothesis, we obtain that the number of consistent decision-mappings defined on $\{X_1, \ldots, X_{i+1}\}$ is

$$2^{|\cup\{\Delta(X_t):t=1,\ldots,i\}|}2^{|W|} = 2^{|\cup\{\Delta(X_t):t=1,\ldots,i+1\}|}$$

which completes the proof.

Remark 1. In particular, if $m = M \setminus R$, then from our Theorem it follows that the number of the maximal consistent decision-mappings is $2^{|O|}$. This shows that there is a strong relationship between the maximal consistent decision-mappings and the subsets of O. Indeed, it can be proved that mapping γ defined by $\gamma(\delta) = op(\delta)$ is a one-to-one mapping of $\Omega'_{\mathbf{M}}$ onto $\wp(O)$ where $\wp(O)$ denotes the set of all subsets of O.

Regarding the relationship between the maximal decision-mappings and the feasible solutions, let us define mapping ρ in the following way. For any $(m, o) \in S(\mathbf{M})$, let $\rho(m, o) = \delta$ where δ is defined by

$$\delta(X) = \{u : u = (\alpha, \beta) \in o \,\&\, X \in \beta\}$$

for all $X \in M \setminus R$. It can be easily proved that ρ is a one-to-one mapping of $S(\mathbf{M})$ into $\Omega'_{\mathbf{M}}$. Therefore, $2^{|O|}$ is a trivial upper bound for $|S(\mathbf{M})|$. Taking into account property (A2), this bound can be improved as follows.

3 Bound calculation

Let $(m,o) \in S(\mathrm{M})$ be an arbitrary feasible solution and $\rho(m,o) = \delta$. Then, (A2) implies the following inclusion:

$$(ii) \qquad mat^{in}(op(\delta)) \subseteq mat^{out}(op(\delta)) \cup R.$$

Indeed, if $X \in mat^{in}(op(\delta))$, then there exists a $u = (\alpha, \beta) \in op(\delta)$ with $X \in \alpha$. By the definition of δ, $u \in o$, and thus, $X \in m$ from the definition of the P-graph. Now, (A2) implies that $X \in mat^{out}(op(\delta)) \cup R$, i.e., inclusion (ii) must hold. Consequently, the number of the maximal consistent decision-mappings satisfying (ii) is not less than $|S(\mathrm{M})|$.

Now, we are going to determine the number of the maximal consistent decision-mappings satisfying (ii). It can be done by the Inclusion-Exclusion Formula. For this purpose, let us denote by (M, O) the P-graph of PNS problem under consideration and let $O = \{u_1, \ldots, u_n\}$ and $M = \{X_1, \ldots, X_k\}$. Furthermore, let $O(X_j)$ denote the set, $\{u : u = (\alpha, \beta) \in O \ \& \ X_j \in \alpha\}$, for all $X_j \in M$. Let $j \in \{1, \ldots, k\}$ be an arbitrary index and let us define set A_j by

$$A_j = \{\delta : \delta \in \Omega'_{\mathrm{M}} \ \& \ X_j \in mat^{in}(op(\delta)) \setminus (mat^{out}(op(\delta)) \cup R)\}.$$

Then, (ii) is not satisfied by $\delta \in A_j$ and the reason is that $X_j \in mat^{in}(op(\delta))$ and $X_j \notin mat^{out}(op(\delta)) \cup R$. For every $\emptyset \neq I \subseteq \{1, \ldots, k\}$, let us define set A_I by $A_I = \cap_{i \in I} A_i$, and in particular, let $A_\emptyset = \Omega'_{\mathrm{M}}$. Then, the required number is

$$|\Omega'_{\mathrm{M}} \setminus (A_1 \cup A_2 \cup \ldots \cup A_k)| = \Sigma_{I \subseteq \{1, \ldots, k\}} (-1)^{|I|} \cdot |A_I| \,.$$

Obviously, if $I = \{i_1, \ldots, i_l\}$, then

$$A_I = \{\delta : \delta \in \Omega'_{\mathrm{M}} \ \& \ \{X_{i_1}, \ldots, X_{i_l}\} \subseteq mat^{in}(op(\delta)) \setminus (mat^{out}(op(\delta)) \cup R)\}$$

Remark 2. It is worth noting that the bound presented above is independent of the set of the required products. It is valid under arbitrary $P \subseteq M \setminus R$.

Unfortunately, to count $|A_I|$ is a difficult problem. In general case, we have to cover $\{X_{i_1}, \ldots, X_{i_l}\}$ with such a system, $\alpha_{j_1}, \ldots, \alpha_{j_s}$ for which there are operating units $(\alpha_{j_t}, \beta_{j_t}) \in O$, $t = 1, \ldots, s$, with $\{X_{i_1}, \ldots, X_{i_l}\} \cap \beta_{j_t} = \emptyset$, $t = 1, \ldots, s$, and $|A_I|$ is equal to the number of the such covering systems. The determination of $|A_I|$ is easier if we restrict ourselves to special classes of PNS problems. An interesting special case is the class containing separator type operating units, i.e., $|\alpha| = 1$ is valid for all $u = (\alpha, \beta) \in O$. In what follows, we deal with this class.

Let us consider set $I = \{i_1, \ldots, i_l\}$ again. Let $O^*(X_{i_j}) = O(X_{i_j}) \setminus (\cup_{i \in I} \Delta\,(X_i))$. Then, $O^*(X_{i_j})$ is the set of operating units such that they do not produce any material from $\{X_t : t \in I\}$ and each of them has X_{i_j} as input material. Now, it is easy to check that

$$|A_I| = \left(\prod_{t=1}^{l} \left(2^{|O^*(X_{i_t})|} - 1 \right) \right) \cdot 2^{|O \setminus (\cup\{\Delta(X_i) : i \in I\}) \setminus (\cup\{O(X_i) : i \in I\})|}$$

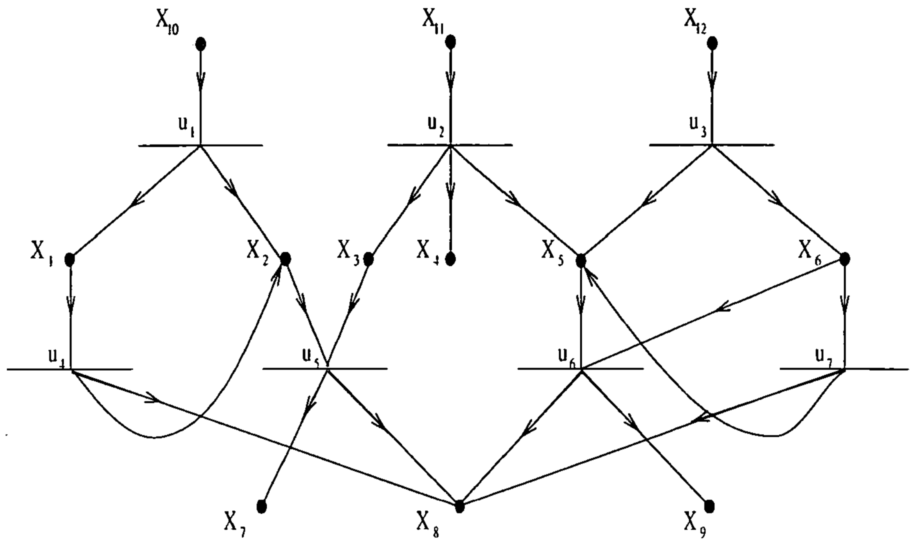

Figure 1: P-graph of the example.

4 Illustration

For illustrating the calculation of the bound in general case, let us consider the following example. Let $M = \{X_1, \ldots, X_{12}\}$, $O = \{u_1, \ldots, u_7\}$, $P = \{X_8\}$, and $R = \{X_{10}, X_{11}, X_{12}\}$. The input and output materials of the operating units are given in Table 1 and the corresponding P-graph is shown in Figure 1.

Table 1: Operating units

unit	inputs	outputs
u_1	X_{10}	X_1, X_2
u_2	X_{11}	X_3, X_4, X_5
u_3	X_{12}	X_5, X_6
u_4	X_1	X_2, X_8
u_5	X_2, X_3	X_7, X_8
u_6	X_5, X_6	X_8, X_9
u_7	X_6	X_5, X_8

Using the relationship between the maximal consistent decision-mappings and the subsets of O provided by Remark 1, set A_1 contains δ if and only if $op(\delta)$ satisfies the following properties: $u_1 \notin op(\delta)$ and $u_4 \in op(\delta)$. The number of such maximal consistent decision-mappings is 2^5. Therefore, $|A_1| = 2^5$. In a similar way, we obtain that $|A_2| = 2^4$, $|A_3| = 2^5$, $|A_5| = 2^3$, $|A_6| = 3 \cdot 2^4$, and $|A_j| = 0$ for the remaining indices. Consequently,

$$\sum_{I \subseteq \{1,\ldots,k\} \;\&\; |I|=1} |A_I| = 136.$$

Regarding the subsets of two elements, $A_{\{1,2\}}$ contains δ if and only if $u_1, u_4 \notin op(\delta)$ and $u_4, u_5 \in op(\delta)$, and thus, $A_{\{1,2\}} = \emptyset$. Similarly, $A_{\{1,3\}} = 2^3$ since $A_{\{1,3\}}$ contains δ if and only if $u_1, u_2 \notin op(\delta)$ and $u_4, u_5 \in op(\delta)$. Determining the corresponding values for the all subsets of two elements and summarizing, we obtain that

$$\sum_{I \subseteq \{1,\ldots,k\} \;\&\; |I|=2} |A_I| = 60.$$

Continuing the procedure, we obtain 12 for the subsets of three elements. Finally, it can be seen that $|A_I| = 0$ if $|I| > 3$. Consequently, the required number is

$$2^7 - 136 + 60 - 12 = 40.$$

We note that $|\Omega'_{\mathrm{M}}| = 128$ and $|S(\mathrm{M})| = 19$ in this example.

References

[1] Z. Blázsik and B. Imreh, A note on connection between PNS and set covering problems, *Acta Cybernetica* **12** (1996), 309-312.

[2] C. A. Floudas and I. E. Grossmann, Algorithmic Approaches to Process Synthesis: Logic and Global Optimization, AiChE Symposium Series No. 304, **91** (Eds: L. T. Biegler and M. F. Doherly), (1995), 198-221. .

[3] F. Friedler, L. T. Fan and B. Imreh, Process Network Synthesis: Problem Definition, *Networks*, to appear.

[4] F. Friedler, K. Tarján, Y. W. Huang and L. T. Fan, Combinatorial Structure of Process Network Synthesis, Sixth SIAM Conference on Discrete Mathematics, Vancouver, Canada, 1992.

[5] F. Friedler, K. Tarján, Y.W. Huang and L.T. Fan, Graph-Theoretic Approach to Process Synthesis: Axioms and Theorems, *Chem. Eng. Sci.*, **47**(8) (1992), 1973-1988.

[6] F. Friedler, J. B. Varga and L. T. Fan, Decision-mappings: a tool for consistent and complete decisions in process synthesis, *Chem. Eng. Sci.*, **50**(11) (1995), 1755-1768.

[7] F. Friedler, J. B. Varga, E. Fehér and L. T. Fan, Combinatorially Accerelated Branch-and -Bound Method for Solving the MIP Model of Process Network Synthesis, presented at the International Conference on State of the Art in Global Optimization: Computational Methods and Applications, Princeton University, Princeton, NJ, U.S.A., April 28-30, 1995; also to be published in Nonconvex Optimization and its Applications, Kluwer Academic Publishers, Norwell, MA, U.S.A. (in press).

[8] I. E. Grossmann, V. T. Voudouris and O. Ghattas, Mixed-Integer Linear Programming Reformulations for Some Nonlinear Discrete Design Optimization Problems, In: Recent Advances in Global Optimization (Eds: C. A. Floudas and P. M. Pardalos) Princeton University Press, New Jersey, 1992.

[9] B. Imreh, F. Friedler and L. T. Fan, An Algorithm for Improving the Bounding Procedure in Solving Process Network Synthesis by a Branch-and-Bound Method, Developments in Global Optimization (Eds: I. M. Bonze, T. Csendes, R. Horst, P. M. Pardalos), Kluwer Academic Publishers, 1997, 315-348.

Received August, 1997