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Pseudo-Hamiltonian Graphs

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Abstract

A pseudo-h-hamiltonian cycle in a graph is a closed walk that visits every vertex exactly h times. We present a variety of combinatorial and algorithmic results on pseudo-h-hamiltonian cycles.

First, we show that deciding whether a graph is pseudo-h-hamiltonian is NP-complete for any given $h \geq 1$. Surprisingly, deciding whether there exists an $h \geq 1$ such that the graph is pseudo-h-hamiltonian, can be done in polynomial time. We also present sufficient conditions for pseudo-h-hamiltonicity that are based on stable sets and on toughness. Moreover, we investigate the computational complexity of finding pseudo-h-hamiltonian cycles on special graph classes like bipartite graphs, split graphs, planar graphs, cocomparability graphs; in doing this, we establish a precise separating line between easy and difficult cases of this problem.

1 Introduction

For an integer $h \geq 1$, we shall say that an undirected graph $G = (V, E)$ is *pseudo-h-hamiltonian* if there exists a circular sequence of $h \cdot |V|$ vertices such that

- every vertex of G appears precisely h times in the sequence, and
- any two consecutive vertices in the sequence are adjacent in G.

A sequence with these properties will be termed a *pseudo-h-hamiltonian cycle*. In this sense, *pseudo-1-hamiltonian* corresponds to the standard notion *hamiltonian*, and a *pseudo-1-hamiltonian cycle* is just a *hamiltonian cycle*. The *pseudo-hamiltonicity number* ph(G) of the graph G, is the smallest integer $h \geq 1$ for which G is pseudo-h-hamiltonian; in case no such h exists, ph$(G) = \infty$. A graph G with finite ph(G) is called *pseudo-hamiltonian*. Pseudo-h-hamiltonicity is a non-trivial graph property. E.g. for every $h \geq 2$, the graph G_h that results from glueing together h triangles at one of their vertices, is pseudo-h-hamiltonian but it is not pseudo-$(h-1)$-hamiltonian.

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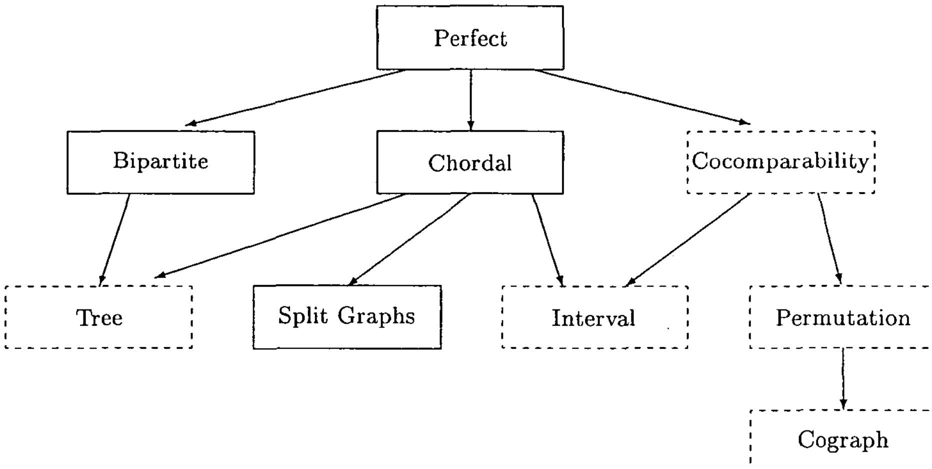

Figure 1: Complexity results for some of the treated graph classes. NP-complete problems have a solid frame, polynomially solvable problems have a dashed frame.

Results of this paper. The problem of deciding whether a given graph is hamiltonian is NP-complete. Hence, it is not surprising at all that for each fixed value of $h \geq 1$, the problem of deciding whether $\mathrm{ph}(G) \leq h$ holds for a given graph G is also NP-complete. However, if we just ask whether $\mathrm{ph}(G) < \infty$, i.e. whether there exists some value of h for which G is pseudo-h-hamiltonian, then we can answer this question in polynomial time (and this is perhaps surprising). This polynomial time result is based on the close relationship of pseudo-hamiltonian graphs with *regularizable* graphs (cf. Section 2).

We also provide a nice and simple characterization of pseudo-hamiltonian graphs that is based on the stable sets of vertices of the graph. We show that every pseudo-hamiltonian graph G must be $1/\mathrm{ph}(G)$-tough, and that every 1-tough graph is pseudo-hamiltonian. The square of a connected graph is always pseudo-hamiltonian. For d-regular graphs with $d \geq 3$, we derive a tight result of the following form: There exists a threshold $\tau(d)$ such that for $h < \tau(d)$, it is NP-complete to decide whether a d-regular graph is pseudo-h-hamiltonian, whereas for every $h \geq \tau(d)$, a d-regular graph automatically is pseudo-h-hamiltonian. Hence, the computational complexity of deciding pseudo-h-hamiltonicity of regular graphs jumps at $\tau(d)$ from trivial immediately to NP-complete.

Finally, we will investigate the computational complexity of computing $\mathrm{ph}(G)$ on many well-known special graph classes, like bipartite graphs, split graphs, partial k-trees, interval graphs, planar graphs etc. Figure 1 summarizes some of our results together with some of their implications for special graph classes. Directed arcs represent containment of the lower graph class in the upper graph class. For classes with a solid frame, the computation of ph(G) is NP-complete, and for classes with a dashed frame, this problem is polynomial time solvable (for exact definitions of all these graph classes cf. Johnson [12]). Note that the results for trees, bipartite graphs, split graphs and cocomparability graphs imply all the other results in Figure 1.

Organization of the paper. Section 2 investigates the connections between pseudo-hamiltonicity and regularizable graphs, and it states several general complexity results. Section 3 relates pseudo-hamiltonicity to stable sets, to connectivity and to toughness. Section 4 derives the complexity threshold for d-regular graphs, and Section 5 deals with squares of graphs. Finally, Section 6 collects the complexity results for the special graph classes.

Notation and conventions. Throughout this paper, we only consider undirected graphs. All graphs have at least three vertices. For convenience we often write $G - W$ instead of $G(V - W)$, the graph that results from removing the vertices in W together with all incident edges from G. For a set $W \subseteq V$, we denote by $N(W)$ the set of all vertices outside W which are adjacent to vertices from W. A stable set is a set of pairwise non-adjacent vertices. A stable set S is maximal if there is no stable set S' which properly contains S. The stability number $\alpha(G)$ is the size of a largest stable set in G.

2 Complexity aspects of pseudo-hamiltonicity

In this section, we give several characterizations of pseudo-hamiltonian graphs that are based on regularizable graphs. These characterizations imply that one can decide in polynomial time whether ph$(G) < \infty$. On the other hand, we will show that for every fixed integer $h \geq 1$ it is NP-complete to decide whether ph$(G) \leq h$.

A graph $G = (V, E)$ is called *regularizable* (see Berge [2, 3]), if for each edge $e \in E$ there is a positive integer $m(e)$ such that the multigraph which arises from G by replacing every edge e by $m(e)$ parallel edges is a regular graph. A useful characterization of regularizable graphs can be found in Berge [2].

Proposition 2.1 *(Berge [2])*
A connected graph $G = (V, E)$ is regularizable if and only if one of the following two statements holds

(a) G is elementary bipartite
(i.e. G is bipartite, connected and every edge of G appears in a perfect matching);

(b) G is 2-bicritical
(i.e. $|N(S)| > |S|$ holds for every stable set $S \subseteq V$). □

Regularizable graphs are related to pseudo-hamiltonian graphs as follows.

Lemma 2.2 *A graph G is pseudo-hamiltonian if and only if G has a connected spanning regularizable subgraph.*

Proof. (Only if). Clearly, in a pseudo-h-hamiltonian cycle (considered as a multigraph) each vertex has degree $2h$. Hence, the *skeleton* of a pseudo-h-hamiltonian cycle (that is, the simple graph arising from replacing parallel edges by simple edges) of a graph G constitutes a regularizable subgraph of G which, additionally, is connected and contains all the vertices of G.

(If). Conversely, assume that a graph G has a connected spanning regularizable subgraph H. Let H^* denote the associated regular multigraph, say of degree $2h$ (if the degree of the regular multigraph is odd, multiply every number $m(e)$ by two). Clearly, H^* has an Eulerian cycle. This Eulerian cycle corresponds to a pseudo-h-hamiltonian cycle in G. □

A graph has a *perfect 2-matching* if one can assign weights 0, 1 or 2 to its edges in such a way that for each vertex, the sum of the weights of the incident edges is equal to 2. The following characterization of regularizable graphs can be found in the book by Lovász and Plummer [13].

Proposition 2.3 *(Lovász and Plummer [13])*
A graph $G = (V, E)$ is regularizable if and only if for each edge $e \in E$ there exists a perfect 2-matching of G in which e has weight 1 or 2. □

Proposition 2.3 has several important consequences.

Corollary 2.4 *(i) For any integer h with $1 \leq h < \mathrm{ph}(G)$, graph G does not possess a pseudo-h-hamiltonian cycle. (ii) For any integer $h \geq \mathrm{ph}(G)$, graph G does possess a pseudo-h-hamiltonian cycle.*

Proof. Statement (i) trivially follows from the definition of $\mathrm{ph}(G)$. In order to prove (ii), we show that if a graph has a pseudo-h-hamiltonian cycle then it also has a pseudo-$(h+1)$-hamiltonian cycle: Let C be a pseudo-h-hamiltonian cycle in G. Then the skeleton of C is regularizable, and consequently possesses a perfect 2-matching. If one adds this perfect 2-matching to the $2h$-regular multigraph that corresponds to C, one gets a $(2h+2)$-regular multigraph that corresponds to a pseudo-$(h+1)$-hamiltonian cycle. □

Proposition 2.3 together with Lemma 2.2 also allows us to construct an algorithm to decide efficiently whether a graph is pseudo-hamiltonian (or, equivalently, to decide whether a graph has a connected spanning regularizable subgraph). The algorithm repeatedly runs through all the edges of the graph and deletes all those edges which do not allow a perfect 2-matching with the desired property. If the remaining graph is disconnected then G is not pseudo-hamiltonian. Otherwise, one obtains a connected spanning regularizable subgraph of G, i.e. G is pseudo-hamiltonian.

Algorithm PSEUDO-HAMILTON(G)

1. UNCHECKED:= E; E^*:= E;
2. **While** UNCHECKED $\neq \emptyset$ **do**
   - **Pick** an arbitrary edge $e \in$UNCHECKED;
   - **Check** whether the graph (V, E^*) possesses a perfect 2-matching in which edge e has weight 1 or 2;
   - **If** there is no such perfect 2-matching **then** E^*:= $E^* - \{e\}$;
   - UNCHECKED:= UNCHECKED$-\{e\}$;
3. **If** the graph (V, E^*) is connected **then return** 'yes' **else return** 'no'.

Since perfect 2-matchings can be found in polynomial time (cf. Lovász and Plummer [13]), the whole algorithm can be implemented to run in polynomial time.

Theorem 2.5 *It can be decided in polynomial time, whether* $\text{ph}(G) < \infty$ *holds for a given graph* G. □

In strong contrast to Theorem 2.5, it is NP-complete to compute $\text{ph}(G)$ exactly.

Theorem 2.6 *For every fixed value* $h \geq 1$, *the problem of deciding whether* $\text{ph}(G) \leq h$ *holds for a given graph* G *is NP-complete.*

Proof. It is well known that deciding pseudo-1-hamiltonicity (i.e. standard hamiltonicity) of a graph is NP-complete. Let $h \geq 2$ be some fixed integer. Consider some undirected graph $G' = (V', E')$, and construct another graph $G = (V, E)$ from it as follows: V contains the vertices in V' together with $3(h-1)|V'|$ new vertices. For every vertex $v \in V'$, there are $3h - 3$ new vertices that are called a_v^i, b_v^i, and c_v^i, where $i = 1, \ldots, h-1$. The edge set E contains all edges in E' together with $4(h-1)|V'|$ new edges. For every vertex $v \in V'$, there are $4h - 4$ new edges (v, a_v^i), (a_v^i, b_v^i), (b_v^i, c_v^i), and (c_v^i, a_v^i), where $i = 1, \ldots, h-1$. We claim that the constructed graph G possesses a pseudo-h-hamiltonian cycle if and only if the original graph G' possesses a hamiltonian cycle.

(Only if). Assume that G possesses a pseudo-h-hamiltonian cycle C. Consider for arbitrary $v \in V'$ and $1 \leq i \leq h-1$ the connected component consisting of a_v^i, b_v^i, and c_v^i. The cycle C can visit and leave this component only via the edge (v, a_v^i), and this edge must be used an even number of times. Hence, C uses at least $2h - 2$ edges incident to v just for visiting the $(h-1)$ attached components. There remain only two edges that can connect v to other vertices in V', and it is easy to see that these pairs of edges taken over all vertices in V' correspond to a hamiltonian cycle in G'.

(If). Now assume that G' possesses a hamiltonian cycle. Construct a multigraph with vertex set V as follows: The multigraph contains all edges that are used by the hamiltonian cycle. Moreover, it contains for every $v \in V'$ and for every i, $1 \leq i \leq h-1$, two copies of the edge (v, a_v^i), $h-1$ copies of the edge (a_v^i, b_v^i), $h+1$ copies of the edge (b_v^i, c_v^i), and $h-1$ copies of the edge (c_v^i, a_v^i). The resulting multigraph is connected and $2h$-regular. Hence, it contains an Eulerian cycle that corresponds to a pseudo-h-hamiltonian cycle in a natural way. □

Question 2.7 *What can be said about approximating* $\mathrm{ph}(G)$*? Can one always find in polynomial time a, say, pseudo-*$2\mathrm{ph}(G)$*-hamiltonian cycle?*

3 Stable sets, connectivity and toughness

This section discusses the relationship of pseudo-hamiltonicity with the structure of stable subsets, with the connectivity of a graph, and with the toughness of a graph. First, consider the following two conditions (C1) and (C2) on a graph $G = (V, E)$.

(C1) $|N(S)| \geq |S|$ holds for every maximal stable set $S \subseteq V$.

(C2) $|N(S)| > |S|$ holds for every non-maximal stable set $S \subseteq V$.

Lemma 3.1 *If a graph $G = (V, E)$ is pseudo-hamiltonian, then it fulfills the conditions (C1) and (C2).*

Proof. Consider a pseudo-h-hamiltonian cycle C and let S be a stable set in G. Every vertex from S appears h times in C. Since S is stable, each vertex from S must be followed by a vertex from $N(S)$. Hence the set $N(S)$ is visited at least $h \cdot |S|$ times. Since each vertex from $N(S)$ also appears h times in C we obtain

$$|N(S)| \geq |S|. \tag{1}$$

Now assume that $|N(S)| = |S|$. Then vertices from S and from $N(S)$ must alternate in C, and it is not possible to visit any vertex from $V - S - N(S)$. This implies that $V = S \cup N(S)$, or equivalently, that S is a maximal stable set. □

Corollary 3.2 *If the graph $G = (V, E)$ with $|V| \geq 3$ vertices is pseudo-hamiltonian then the following holds:*

(a) G has no vertices of degree one.

(b) $\alpha(G) \leq \frac{1}{2}|V|$. □

We can use the results on regularizable graphs (cf. Section 2) in order to show that, for a connected graph, the conditions (C1) and (C2) are also sufficient for the existence of a pseudo-hamiltonian cycle.

Lemma 3.3 *If a connected graph $G = (V, E)$ fulfills conditions (C1) and (C2), then it is pseudo-hamiltonian.*

Proof. If $|N(S)| > |S|$ holds for every stable set $S \subseteq V$ then G is 2-bicritical and, by Proposition 2.1, also regularizable. Since G is connected, Lemma 2.2 implies that in this case G is pseudo-hamiltonian.

Otherwise, there exists a stable set S with $|N(S)| = |S|$. Then by condition (C1), S is maximal and $V = S \cup N(S)$ holds. Let H denote the spanning subgraph of G which arises from deleting all edges between vertices from $N(S)$. We show that H is elementary bipartite. Then, again by Proposition 2.1, the subgraph H is regularizable and, since H is also connected, Lemma 2.2 implies that G is pseudo-hamiltonian.

By construction, the graph H is bipartite. H is connected, since otherwise we can easily find a proper subset $S' \subset S$ with $|N(S')| \leq |S'|$ in contradiction to the assumption. Let (s,t) be an arbitrary edge in H with $s \in S$. In $H - \{s,t\}$ we have $|N(S')| \geq |S'|$ for each set $S' \subseteq S - \{s\}$ (note that S' is not maximal stable in G). It is well known that this condition implies the existence of a perfect matching in $H - \{s,t\}$ (cf. e.g. Lovász and Plummer [13]). Hence there is a perfect matching in H containing the edge (s,t). □

Every hamiltonian graph must be 2-connected. However, it is easy to see that this is not a necessary condition for a graph to be pseudo-h-hamiltonian for some $h \geq 2$. On the other side one may ask whether there exists a number k such that every k-connected graph is also pseudo-hamiltonian. The following example shows that this is not true in general.

Example 3.4 *Consider the complete bipartite graph $K_{k+1,k}$, i.e. the graph consisting of two stable sets S and S' of cardinality $k+1$ and k, respectively, where any two vertices from S and S' are adjacent. By deleting fewer than k vertices, we leave at least one node in the stable set S and at least one node in the stable set S'. Hence, this graph is k-connected. However, since $|N(S)| = k < k+1 = |S|$, we conclude from Lemma 3.1 that the graph is not pseudo-hamiltonian.*

Chvátal [7] defines the *toughness* $t(G)$ of a graph G (where G is not a complete graph) by

$$t(G) \;=\; \min \frac{|W|}{c(G-W)}, \tag{2}$$

where W is a cutset of G and $c(G-W)$ denotes the number of connected components of the graph $G - W$. It is well known that a hamiltonian graph has toughness at least 1. As an extension of this result we obtain:

Lemma 3.5 *If G is pseudo-h-hamiltonian, then $t(G) \geq \frac{1}{h}$.*

Proof. Let W^* be a cutset of G with $t(G) = |W^*|/c(G - W^*)$. Each path between two vertices of different connected components of $G - W^*$ contains vertices from W^*. Hence, in a pseudo-h-hamiltonian cycle of G there appears at least $c(G - W^*)$ times a vertex from W^*, i.e. each vertex from W^* appears at least $c(G - W^*)/|W^*|$ times. This implies $h \geq 1/t(G)$ and the correctness of the claim. □

It is known (cf. Chvátal [7]) that there are graphs with toughness 1 which are not hamiltonian. Similarly, the converse of Lemma 3.5 is not always true for $h \geq 2$. The complete bipartite graph $K_{3,2}$ has toughness $t(K_{3,2}) = 2/3 \geq 1/h$. However, as argued in Example 3.4 above, this graph is not pseudo-h-hamiltonian.

Another sufficient condition for pseudo-hamiltonicity relies on the toughness of the graph.

Lemma 3.6 *(i) Any graph G with $t(G) \geq 1$ is pseudo-hamiltonian. (ii) For every $\varepsilon > 0$, there exists a graph G with $t(G) \geq 1 - \varepsilon$ that is not pseudo-hamiltonian.*

Proof. Consider a graph G with toughness at least 1. Clearly, G is connected. We will show that G fulfills the conditions (C1) and (C2), and then Lemma 3.3 implies statement (i).

Let S be a maximal stable set in G and assume that $|N(S)| < |S|$ holds. With $W := N(S)$, we obtain $c(G - W) > |W|$ as the vertices of S form the connected components of $G - W$. Hence $t(G) < 1$, in contradiction to the assumption.

Let S be a non-maximal stable set in G and assume that $|N(S)| \leq |S|$. Define again $W := N(S)$. Then the vertices of S are again connected components of $G - W$, and since S is not maximal there is at least one further component. Hence $c(G - W) > |W|$ holds, which implies that $t(G) < 1$.

In order to prove (ii), consider the complete bipartite graphs $K_{k+1,k}$ from Example 3.4: $K_{k+1,k}$ has toughness $k/(k+1)$. As k tends to infinity, this expression tends to one. □

4 Regular graphs

In this section, we discuss the problem of deciding whether a given d-regular graph possesses a pseudo-h-hamiltonian cycle. We will show that for every d, there is a precise threshold for h where the computational complexity of recognizing pseudo-h-hamiltonian d-regular graphs jumps from NP-complete to trivial.

Lemma 4.1 *(i) For odd $d \geq 3$, every connected d-regular graph G fulfills* $\mathrm{ph}(G) \leq d$. *(ii) For even $d \geq 4$, every connected d-regular graph G fulfills* $\mathrm{ph}(G) \leq d/2$.

Proof. For even d, graph G itself is Eulerian and the Eulerian cycle yields a pseudo-$d/2$-hamiltonian cycle. For odd d, the multigraph that contains two copies of every edge in G is Eulerian and thus yields a pseudo-d-hamiltonian cycle. □

Lemma 4.2 *(i) For odd $d \geq 3$, it is NP-complete to decide whether* $\mathrm{ph}(G) \leq d - 1$ *holds for a d-regular graph G. (ii) For even $d \geq 4$, it is NP-complete to decide whether* $\mathrm{ph}(G) \leq d/2 - 1$ *holds for a d-regular graph G.*

Proof. We only prove (i). The proof of (ii) can be done by analogous (somewhat tedious) arguments.

For every odd $d \geq 3$, the proof of (i) is based on the following auxiliary graph H_d: H_d has $2d - 1$ vertices that are divided into three parts X, Y and Z. Part X consists of a single vertex x, parts $Y = \{y_1, \ldots, y_{d-1}\}$ and $Z = \{z_1, \ldots, z_{d-1}\}$ both contain $d - 1$ vertices. There is an edge between x and every vertex in Y, and there is an edge between every vertex in Y and every vertex in Z. Moreover, the vertices in Z are connected to each other by a perfect matching in such a way that z_1 and z_2 are matched with each other. This completes the description of H_d. Note that in H_d, vertex x has degree $d - 1$ and all vertices in $Y \cup Z$ have degree

d. Moreover, we will use the following connected multigraph $M(H_d)$: $M(H_d)$ has the same vertex set as H_d. Vertex x is connected by a single edge to y_1 and y_2, respectively, and by two edges to each vertex in $Y - \{y_1, y_2\}$. For $1 \le j \le 2$, y_j is connected by $2d-3$ edges to z_j, and for $3 \le j \le d-1$, y_j is connected by $2d-4$ edges to z_j. Finally, there is one edge that connects z_1 to z_2, and there are two copies of every other edge in the matching over Z. Note that in the resulting graph $M(H_d)$, vertex x has degree $2d-4$ and all vertices in $Y \cup Z$ have degree $2d-2$.

The NP-completeness proof for result (i) is done by a reduction from the NP-complete hamiltonian cycle problem in cubic graphs (cf. Garey and Johnson [11]). Consider an instance $G' = (V', E')$ of this problem, and construct a d-regular graph $G = (V, E)$ from G' as follows:

- For every $v \in V'$, introduce a corresponding vertex v^* in V. Moreover, introduce $d-3$ pairwise disjoint copies of H_d. The x-vertex of every such copy is connected to v^*.
- For every edge $(u, v) \in E'$, introduce two new vertices $a_{u,v}$ and $a_{v,u}$ together with the three edges $(u^*, a_{u,v})$, $(a_{u,v}, a_{v,u})$ and $(a_{v,u}, v^*)$, i.e. the vertices $a_{u,v}$ and $a_{v,u}$ essentially subdivide the original edge (u, v) into three sub-edges.
- For every new vertex $a_{u,v}$, create $d-2$ pairwise disjoint copies of H_d and connect the x-vertex of every copy to $a_{u,v}$.

It is easy to verify that the resulting graph G is d-regular (since in H_d, vertex x has degree $d-1$ and all other vertices have degree d). We claim that G possesses a pseudo-$(d-1)$-hamiltonian cycle if and only if G' possesses a hamiltonian cycle.

(If). Assume that G' possesses a hamiltonian cycle. Construct from this hamiltonian cycle a $(2d-2)$-regular multigraph M^* as follows: For every copy of H_d in G, introduce the corresponding edges of $M(H_d)$ in M^*, together with two edges that connect the x-vertex to that vertex to which the copy has been attached. For every edge (u, v) that is used by the hamiltonian cycle, introduce the three edges $(u^*, a_{u,v})$, $(a_{u,v}, a_{v,u})$ and $(a_{v,u}, v^*)$ in M^*. For every edge (u, v) that is not used by the hamiltonian cycle, introduce two copies of $(u^*, a_{u,v})$ and two copies of $(a_{v,u}, v^*)$ in M^*. The resulting multigraph is $(2d-2)$-regular, is connected (as it simulates the hamiltonian cycle in G'), and it is spanning. Hence, the corresponding Eulerian cycle in G yields a pseudo-$(d-1)$-hamiltonian cycle for G.

(Only if). Now assume that G possesses a pseudo-$(d-1)$-hamiltonian cycle C. Then the edges that are traversed by C form a $(2d-2)$-regular connected multigraph M^C. For every copy of H_d in G, the cycle C traverses the edge that connects the x-vertex to the vertex to which the copy has been attached, at least twice and an even number of times. Hence, for every edge $(u, v) \in E'$ the vertex $a_{v,u}$ in M^C is connected by at least $2d-4$ edges to the x-vertices of the attached copies of H_d, and there remain only two edges that can connect $a_{v,u}$ to the rest of the graph. With this it is easy to verify that there remain only two possibilities how the cycle C may traverse the three edges $(u^*, a_{u,v})$, $(a_{u,v}, a_{v,u})$ and $(a_{v,u}, v^*)$ that correspond to some edge $(u, v) \in E'$ in the original graph: Either all three edges are traversed thus resulting multigraph is 4-regular and contains only edges from G^2. Hence, G^2 is pseudo-2-hamiltonian.

(Only if). Now assume that G^2 possesses a pseudo-2-hamiltonian cycle C. The following statements on the structure of C are easy to verify.

1. C traverses every edge (b_v^i, v) with $v \in V_1'$ and $1 \le i \le 4$ exactly once.
2. C traverses every edge (b_v^i, a_v^i) with $v \in V_1'$ and $1 \le i \le 4$ exactly three times.
3. For every $v \in V_1'$, C either traverses exactly one or exactly zero of the edges (a_v^i, a_v^j) with $1 \le i < j \le 4$.
4. C traverses every edge (d_v^i, v) with $v \in V_2'$ and $1 \le i \le 2$ exactly once.
5. C traverses every edge (d_v^i, c_v^i) with $v \in V_2'$ and $1 \le i \le 2$ exactly three times.
6. C traverses every edge (c_v^1, c_v^2) with $v \in V_2'$ exactly once.

Hence, every $v \in V_1'$ is only connected to vertices b_v^i. Every $v \in V_2'$ must be connected by two edges to some vertices a_u^i. Hence, there are exactly $2|V_2'|$ edges between V_2' and the a_u^i with $u \in V_1'$, and a simple counting argument shows that in statement (3) above, the "traverses exactly zero of the edges"-part can never hold. Hence, for every $v \in V_1'$ there exist exactly two edges in C that connect some a_v^i to some vertex $u \in V_2'$. It is straightforward to see that the union of all these edges corresponds to a hamiltonian cycle in G'. □

6 Special graph classes

In this section, we show that deciding whether a graph is pseudo-h-hamiltonian is NP-complete even for some very restricted classes of graphs that possess a strong combinatorial structure. Moreover, we present polynomial time algorithms for other classes of structured graphs.

6.1 Trees and planar graphs

By Corollary 3.2.(a), a pseudo-hamiltonian graph cannot have any vertices of degree one. Hence, $\text{ph}(T) = \infty$ for any tree T.

If we start the construction in the proof of Theorem 2.6 with a planar graph G', then the constructed graph G is also planar. Since deciding hamiltonicity of planar graphs is NP-complete [11], we conclude that for every $h \ge 1$ it is NP-complete to decide whether a planar graph is pseudo-h-hamiltonian.

6.2 Partial k-trees

The class of *partial k-trees* is a well-known generalization of ordinary trees (see e.g. the survey articles by Bodlaender [4, 5, 6] and by van Leeuwen [14]). It is known that series-parallel graphs and outerplanar graphs are partial 2-trees and that Halin graphs are partial 3-trees. Large classes of algorithmic problems can be solved in polynomial time on partial k-trees if k is constant. Essentially, each graph problem that is expressible in the Monadic Second Order Logic (MSOL) is solvable in linear time on partial k-trees with constant k (cf. e.g. Arnborg, Lagergren, Seese [1]).

Lemma 6.1 *For every $h \geq 1$ and for every $k \geq 1$, it can be decided in linear time whether a given partial k-tree is pseudo-h-hamiltonian.*

Proof. We only show the statement for $h = 2$; the other cases can be settled analogously. For a given graph $G = (V, E)$, the property of having a connected 4-regular submultigraph can be expressed in MSOL as follows:

1. There exist three pairwise disjoint subsets E_1, E_2 and E_3 of E
2. Every vertex is either incident to (i) four edges from E_1, or to (ii) two edges from E_1 and one edge from E_2, or to (iii) one edge from E_1 and one edge from E_3, or to (iv) two edges from E_2
3. There does not exist a partition of the vertex set V into two non-empty sets V_1 and V_2, such that none of the edges in $E_1 \cup E_2 \cup E_3$ connects V_1 to V_2.

Intuitively speaking, the edges in E_1 (E_2, E_3) occur once (twice, thrice) in the submultigraph. The second condition then takes care of the 4-regularity, and the third condition ensures that the submultigraph is connected. □

6.3 Bipartite graphs and split graphs

Lemma 6.2 *For every integer $h \geq 1$, it is NP-complete to decide whether a bipartite graph is pseudo-h-hamiltonian.*

Proof. It is NP-complete to decide whether a bipartite graph G' is hamiltonian (cf. Garey and Johnson [11]). Consider a bipartite graph $G' = (V', E')$ with bipartition $V' = V_1' \cup V_2'$, and construct from G' another bipartite graph G as follows. For every vertex $v \in V'$, introduce two vertices ℓ_v and r_v in V together with auxiliary vertices a_v^i and b_v^i, $i = 1, \ldots, 2h-2$. In E, there are the edges (ℓ_v, r_v) together with the edges (ℓ_v, a_v^i), (a_v^i, b_v^i), and (b_v^i, r_v) for $i = 1, \ldots, 2h-2$. Moreover, for every edge $(u, v) \in E'$ with $u \in V_1'$ and $v \in V_2'$, we introduce the two edges (ℓ_u, r_v) and (ℓ_v, r_u).

It can be verified that the resulting graph G is also bipartite. Moreover, one can show that G possesses a pseudo-h-hamiltonian cycle if and only if G' possesses a hamiltonian cycle. □

A *split graph* is a graph whose vertex set can be partitioned into two parts such that the subgraph induced by the first part is a clique and the subgraph induced by the second part is a stable set.

Corollary 6.3 *For every integer $h \geq 1$, it is NP-complete to decide whether a split graph is pseudo-h-hamiltonian.*

Proof. In the NP-completeness proof for bipartite graphs in Lemma 6.2, both classes in the bipartition of the constructed graph G are of equal cardinality. Transform G into a split graph G^* by adding all edges between vertices in one part of the bipartition. It is easy to see that a pseudo-h-hamiltonian cycle in G^* can never use these added edges, and hence G^* is pseudo-h-hamiltonian if and only if G' is hamiltonian. □

6.4 Cocomparability graphs

A *comparability graph* is a graph $G = (V, E)$ whose edges are exactly the comparable pairs in a partial order on V. The complementary graph is called a *cocomparability graph*. The class of cocomparability graphs properly contains all cographs, permutation graphs and interval graphs.

Lemma 6.4 *For every integer $h \geq 1$, it can be decided in polynomial time whether a cocomparability graph is pseudo-h-hamiltonian.*

Proof. It is known that a hamiltonian cycle in a cocomparability graph can be found in polynomial time (cf. Deogun and Steiner [9]). Given a cocomparability graph $G = (V, E)$, we construct another cocomparability graph $G' = (V', E')$ as follows. V' contains the vertices in V together with $(h-1)|V|$ new vertices. For every vertex $v \in V$ there are $h-1$ new vertices that are called v^i, where $i = 2, \ldots, h$. For simplicity of notation, let $v^1 := v$. If (u, v) is an edge in E then all edges (u^i, v^j) with $i, j = 1, \ldots, h$ belong to E' (roughly spoken, G' arises from G by replacing each vertex by a stable set of h vertices). It is easy to see that G' is again a cocomparability graph. We show that G has a pseudo-h-hamiltonian cycle if and only if G' has a hamiltonian cycle.

(If). Assume that G' possesses a hamiltonian cycle. We obtain a pseudo-h-hamiltonian cycle in G if each vertex v^i, $i = 2, \ldots, h$, is replaced by the corresponding vertex v.

(Only if). Now assume that G possesses a pseudo-h-hamiltonian cycle $\mathcal{C}$. Each vertex of G appears h times in $\mathcal{C}$. For each $v \in V$ replace $h-1$ copies of v in $\mathcal{C}$ by $v^2, \ldots, v^h$. This yields a *2-factor* of G', i.e. a subgraph of G' such that each vertex has degree 2. If the 2-factor is a cycle then we have a hamiltonian cycle in G' and we are done. Otherwise the 2-factor is a disjoint union of cycles. In this case the following principle allows to reduce the number of cycles: Let $\mathcal{C}_1$ and $\mathcal{C}_2$ denote two disjoint cycles such that v^i belongs to $\mathcal{C}_1$ and v^j belongs to $\mathcal{C}_2$ (it is straightforward to see that such cycles must exist). Let further x be the predecessor of v^i in $\mathcal{C}_1$ and y the predecessor of v^j in $\mathcal{C}_2$. Replace the edges (x, v^i) and (y, v^j) by (x, v^j) and (y, v^i). One obtains a new cycle that contains all vertices from $\mathcal{C}_1$ and $\mathcal{C}_2$. Repeatedly merging cycles in this way finally provides the desired hamiltonian cycle in G'. □

We leave it as an open problem to determine the complexity of computing the pseudo-hamiltonicity number of *asteroidal triple-free* graphs, AT-free graphs for short (cf. Corneil, Olariu, and Stewart [8]). Note that for an AT-free graph G, the graph G' that is constructed in the proof of Lemma 6.4 above is also AT-free. However, the complexity of finding a hamiltonian cycle in AT-free graphs is currently unknown.

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