On directable nondeterministic trapped automata*

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Abstract

A finite automaton is said to be directable if it has an input word, a directing word, which takes it from every state into the same state. For nondeterministic (n.d.) automata, directability can be generalized in several ways. In [8], three such notions, D1-, D2-, and D3-directability, are introduced. In this paper, we introduce the trapped n.d. automata, and for each \( i = 1, 2, 3 \), present lower and upper bounds for the lengths of the shortest Di-directing words of \( n \)-state Di-directable trapped n.d. automata. It turns out that for this special class of n.d. automata, better bounds can be found than for the general case, and some of the obtained bounds are sharp.

1 Introduction

An input word \( w \) is called a directing (or synchronizing) word of an automaton \( A \) if it takes \( A \) from every state to the same state. Directable automata have been studied extensively, we mention only some of the related works (see e.g. [3],[4],[5],[7],[10],[12]). Directable n.d. automata have received less attention. Directability of n.d. automata can be defined in several meaningful ways. The following three nonequivalent definitions are introduced and studied in [8]. An input word \( w \) of an n.d. automaton \( A \) is said to be

(1) D1-directing if it takes \( A \) from every state to the same singleton set,
(2) D2-directing if it takes \( A \) from every state to the same fixed set \( A' \), where \( \emptyset \subseteq A' \subseteq A \), and
(3) D3-directing if there is a state \( c \) such that \( c \notin aw \), for every \( a \in A \).

The D1-directability of complete n.d. automata was investigated by Burkhard [1]. He gave a sharp exponential bound for the lengths of minimum-length D1-directing words of complete n.d. automata. In [6] on games of composing relations over a finite set Goralčík \( it et al.\), in effect, studied D1- and D3-directability and they proved that neither for D1- nor for D3-directing words, the bound can be polynomial

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for n.d. automata. Carpi [2] considered a particular class of n.d. automata, the class of unambiguous n.d. automata, and presented $O(n^3)$ bounds for the lengths of their shortest D1-directing words.

Here we study trapped n.d. automata that have a trap state, i.e., a state which is stable for any input symbol, and present lower and upper bounds for the lengths of their shortest directing words of the three different types.

2 Preliminaries

Throughout this paper $X$ always denotes a finite nonempty alphabet. The set of all finite words over $X$ is denoted by $X^*$ and $A$ denotes the empty word. The length of a word $w \in X^*$ is denoted by $|w|$. For any $p,q \in X^*$, the word $p$ is called a prefix of $q$ if there exists a word $s \in X^*$ such that $ps = q$. For the sake of simplicity, we use the notation $[n]$ for the set $\{1,\ldots,n\}$.

By a nondeterministic (n.d.) automaton we mean a system $A = (A,X)$, where $A$ is a nonempty finite set of states, $X$ is the input alphabet, and each input symbol $x \in X$ is realized as a binary relation $x^A \subseteq A \times A$. For any $a \in A$ and $x \in X$, let $ax^A = \{b : b \in A \text{ and } (a,b) \in x^A\}$. Moreover, for every $B \subseteq A$, we denote by $Bx^A$ the set $\bigcup\{ax^A : a \in B\}$. Now, for any word $w \in X^*$ and $B \subseteq A$, $Bw^A$ can be defined inductively as follows:

1. $B \lambda^A = B$;
2. $Bw^A = (Bp^A)x^A$ for $w = px$, where $p \in X^*$ and $x \in X$.

If $w = x_1 \ldots x_m$ and $a \in A$, then let $aw^A = \{a\}w^A$. This yields that $w^A = x_1^A \ldots x_m^A$. If there is no danger of confusion, then we write simply $aw$ and $Bw$ for $aw^A$ and $Bw^A$, respectively.

An n.d. automaton $A = (A,X)$ is complete if $ax \neq \emptyset$ holds, for all $a \in A$ and $x \in X$. Complete n.d. automata are called c.n.d. automata for short. A state of an n.d. automaton $A$ is called a trap if it is stable for any input symbol, i.e., $ax = \{a\}$, for every input symbol $x$ of $A$. An n.d. automaton is called trapped if it has a trap. Let us denote the class of trapped n.d. automata by $T$. Regarding some recent results on trapped deterministic automata, we refer to the works [9],[11],[12]. Following [8] we define the directability of n.d. automata as follows. Let $A = (A,X)$ be an n.d. automaton. For any word $w \in X^*$, let us consider the following conditions:

(D1) $\exists c \in A)(\forall a \in A)(aw = \{c\})$,
(D2) $\forall a,b \in A)(aw = bw)$,
(D3) $\exists c \in A)(\forall a \in A)(c \in aw)$.

For any $i = 1,2,3$, if $w$ satisfies Di, then $w$ is called a Di-directing word of $A$ and in this case $A$ is said to be Di-directable. Let us denote by $D_i(A)$ the set of Di-directing words of $A$, moreover, let $\text{Dir}(i)$ and $\text{CDir}(i)$ denote the classes of
Di-directable n.d. automata and c.n.d. automata, respectively. Now, we can define the following functions. For any \( i = 1, 2, 3 \) and \( \mathcal{A} = (A, X) \in \text{Dir}(i) \), let
\[
d_i(\mathcal{A}) = \min\{|w| : w \in D_i(\mathcal{A})\},
\]
\[
d_i(n) = \max\{d_i(\mathcal{A}) : \mathcal{A} \in \text{Dir}(i) & |\mathcal{A}| = n\},
\]
\[
cd_i(n) = \max\{d_i(A) : A \in \text{CDir}(i) & |A| = n\}.
\]
The functions \( d_i(n), cd_i(n), i = 1, 2, 3 \), are studied in [8], where lower and upper bounds depending on \( n \) are presented for them. Similar functions can be defined for any class of n.d. automata. For a class \( K \) of n.d. automata, let
\[
d_i^K(n) = \max\{d_i(A) : A \in \text{Dir}(i) \cap K & |A| = n\},
\]
\[
cd_i^K(n) = \max\{d_i(A) : A \in \text{CDir}(i) \cap K & |A| = n\}.
\]
Obviously, \( cd_i^K(n) \leq d_i^K(n) \), for \( i = 1, 2, 3 \).

In what follows, we study the case when the considered class is \( T \), the class of trapped n.d. automata. It is worth noting that if a trapped n.d. automaton is Di-directable, then it has only one trap.

### 3 Directable trapped n.d. automata

First we deal with the D3-directability. We consider D3-directable trapped c.n.d. automata, and using certain deterministic automata, introduced by Rystsov [12], we present an exact bound for this class. Then we study D3-directable trapped n.d. automata and present lower and upper bounds for the lengths of their shortest D3-directing words. For trapped c.n.d. automata the following statement is valid.

**Theorem 1.** For any \( n \geq 1 \), \( cd_3^T(n) = (n - 1)n/2 \).

**Proof.** First we prove that \( (n - 1)n/2 \leq cd_3^T(n) \). This inequality follows from Theorem 6.1 in [12]. Since the proof is short, we recall it for the sake of completeness.

For every \( n > 1 \), let us define the c.n.d. automaton \( B_n = (\{0, 1, \ldots, n - 1\}, \{x_1, \ldots, x_{n-1}\}) \) as follows. Let \( 0x_1 = 1x_1 = \{0\}, \) and \( jx_1 = \{j\}, j = 2, \ldots, n-1 \). Moreover, for all \( 2 \leq k \leq n-1 \) and \( j \in \{0, 1, \ldots, n - 1\} \), let
\[
jx_k = \begin{cases} 
\{j - 1\} & \text{if } j = k, \\
\{j + 1\} & \text{if } j = k - 1, \\
\{j\} & \text{otherwise}.
\end{cases}
\]

Obviously, \( B_n \) is a D3-directable trapped c.n.d. automaton with the trap 0. Let us observe that for any \( j \in \{0, 1, \ldots, n - 1\} \), \( jp \) is a singleton set whenever \( p \in X^* \), moreover, \( jw = \{0\} \) for any D3-directing word of \( B_n \), because 0 is a trap state. Therefore, \( \{0, 1, \ldots, n - 1\}w = \{0\} \), for any \( w \in D_3(B_n) \). Now, let us assign to every nonempty subset \( J \) of states a weight, denoted by \( g(J) \), which is the sum of the numbers contained in \( J \), i.e.,
Then \( g(J) = \sum_{j \in J} j \). Then \( g(\{0, 1, \ldots, n - 1\}) = (n - 1)n/2 \) and for any nonempty subset \( J \) of \( \{0, 1, \ldots, n - 1\} \) and input sign \( x_k, k \in [n - 1] \),

\[
|g(J) - g(Jx_k)| \leq 1.
\]

From these facts it follows that the length of any D3-directing word of \( B_n \) is not less than \( (n - 1)n/2 \), because this word brings the state set of weight \( (n - 1)n/2 \) into the set \( \{0\} \) with weight 0. Hence, \( (n - 1)n/2 \leq d_3(B_n) \). On the other hand, it is easy to check that the word

\[
w = x_1x_2x_3x_2x_1 \ldots x_{n-1}x_{n-2} \ldots x_2x_1
\]

is a D3-directing word of \( B_n \) and \( |w| = (n - 1)n/2 \). Consequently,

\[
d_3(B_n) = (n - 1)n/2.
\]

Since \( B_n \) is a D3-directable trapped c.n.d. automaton of \( n \) states, the equality above implies \( (n - 1)n/2 \leq c_d^F(n) \).

In order to prove that this bound is sharp, we prove that for any D3-directable trapped c.n.d. automaton \( A = (A, X) \) of \( n > 1 \) states, there exists a D3-directing word whose length is not greater than \( (n - 1)n/2 \). To simplify the notation, we assume that \( A = \{0, 1, \ldots, n - 1\} \) and 0 is the trap of \( A \). Since \( A \) is a D3-directable c.n.d. automaton and \( 0x = \{0\} \), for all \( x \in X \), there exists for any state \( j \in A \) a word \( x_1 \ldots x_m \) of minimum-length such that \( 0 \in jx_1 \ldots x_m \). Moreover, there are states \( j_1, \ldots, j_{m-1} \in A \) such that \( j_t \in jx_1 \ldots x_t \) and \( 0 \in j_t x_{t+1} \ldots x_m \), for all \( t = 1, \ldots, m-1 \). Since \( x_1 \ldots x_m \) is a minimum-length word satisfying \( 0 \in jx_1 \ldots x_m \), the states \( j, j_1, \ldots, j_{m-1}, 0 \) must be pairwise different. Therefore, by \( |A| = n \), we obtain \( m \leq n - 1 \). Observe that for any \( 2 \leq t \leq m \), \( x_t \ldots x_m \) is a minimum-length word satisfying \( 0 \in j_{t-1}x_t \ldots x_m \). Based on these observations, by renaming the states, we may suppose that for any state \( j \in A \), there exists a word \( p_j \) such that \( 0 \in jp_j \) and \( |p_j| \leq j \). By using the pairs \( j, p_j, j = 0, \ldots, n - 1 \), we present a procedure for finding a D3-directing word with length, not greater than \( (n - 1)n/2 \).

Initialization. Let \( t = 0, B_0 = \{0\}, p_{i_0} = \lambda, \) and \( R_0 = \{1, 2, \ldots, n - 1\} \).

Iteration.

- **Step 1.** Terminate if \( A = B_t \). Otherwise proceed to Step 2.
- **Step 2.** For each \( j \in R_t \), let \( k_j^{(t)} \) denote the smallest number in the set \( jp_{i_t} \). Select the least element in \( \{k_j^{(t)} : j \in R_t\} \) and denote it by \( i_{t+1} \). Let

\[
B_{t+1} = \{j : j \in A \& 0 \in jp_{i_0} \ldots p_{i_{t+1}}\}.
\]
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and

$$R_{t+1} = \{k_j^{(t)} : j \in A \setminus B_{t+1}\}.$$ 

Increase the value of $t$ by 1 and proceed to the next iteration.

To verify the correctness of the above procedure, we note the following facts.

(i) For any $i_{t+1}$, there exists a $j \in A \setminus B_t$ such that $i_{t+1}$ is an element of the set $j p_{i_0} \ldots p_{i_t}$. Then $B_t \cup \{j\} \subseteq B_{t+1}$, and hence, $B_t \subseteq B_{t+1}$.

(ii) If $j \in A \setminus B_t$, then $0 \not\in j p_{i_0} \ldots p_{i_t}$ yielding $k_j^{(t)} > 0$. Therefore, $R_t$ is a set of positive integers.

(iii) If $A \neq B_t$, then there is a $j \in A \setminus B_t$ with $j p_{i_0} \ldots p_{i_t} \not\in \emptyset$ since $A$ is a c.n.d. automaton, and thus, $R_t \neq \emptyset$. Consequently, $A \neq B_t$ implies $R_t \neq \emptyset$.

From these facts it follows that there exists a positive integer $s \leq n - 1$ such that $A = B_s$. Now, by the definition of $B_s$, we obtain that

$$w = p_{i_0} \ldots p_{i_s}$$

is a D3-directing word of $A$. Let $r_t = |R_t|$, $t = 0, \ldots, s - 1$. From the definition of $R_t$ it follows that

$$n - 1 \geq r_0 > r_1 > \ldots > r_{s-1} > 0.$$ 

On the other hand, since $|R_t| = r_t$, the least number $i_{t+1}$ of $\{k_j^{(t)} : j \in R_t\}$ is not greater than $n - r_t$. This yields that $|p_{i_{t+1}}| \leq n - r_t$, $t = 0, \ldots, s - 1$. Since $|p_{i_0}| = 0$, we obtain that

$$|w| \leq \sum_{t=0}^{s-1} (n - r_t).$$

Let us observe that the numbers $n - r_t$, $t = 0, \ldots, s - 1$ are pairwise different and each of them is contained in the set $[n - 1]$. Therefore, the upper bound of $|w|$ is the sum of some distinct elements of $[n - 1]$. But this sum is not greater than the sum $(n - 1)n/2$ of all the elements of $[n - 1]$. Consequently, $|w| \leq (n - 1)n/2$. If $n = 1$, then the statement is obviously also valid. This completes the proof of Theorem 1. 

For D3-directable trapped n.d. automata, we have the following bounds.

**Theorem 2.** For any $n \geq 2$, $\max\{[n^3 - 1]!,(n - 2)^2 + 1\} \leq d_F^T(n) \leq 2^{n-1} - 1$.

**Proof.** The first member in the lower bound comes from the general case (cf. [8]), where the automata, providing this bound, are trapped automata. The second member in the lower bound can be derived from Černý's well-known examples (cf. [3]) as follows. One can equip Černý's automaton of $n - 1$ states with a trap state and a new input symbol, denoted by $\diamond$ and $z$, respectively. Let $\diamond z = \{\diamond\}$, $0z = \{\diamond\}$,
and \( jz = \emptyset \), for all \( j = 1, \ldots, n - 1 \). Now, redefine the remaining transitions as follows. If \( ax = b \), then let \( ax = \{b\} \) be the new transition. Then we obtain an n.d. automaton of \( n \) states whose shortest D3-directing words are of length \((n - 2)^2 + 1\).

To obtain the upper bound, let us consider an arbitrary D3-directable trapped n.d. automaton \( A = (A, X) \) of \( n (> 1) \) states. Let \( A = \{a_1, \ldots, a_n\} \) and \( a_n \) be the trap of \( A \). Let \( \{a_1, \ldots, a_n\} \) and \( a_n \) be the trap of \( A \). Let \( w = x_1 \ldots x_m \) be a minimum-length D3-directing word of \( A \). Then \( a_n w = \{a_n\} \) and by the D3-directability of \( A \), \( a_n \in ax_1 \ldots x_k \) for every \( k \in [m] \). By using these observations, it is easy to see that if \( S_j = S_l \) for some \( 0 < j < l \leq m \), then \( x_1 \ldots x_j x_{l+1} \ldots x_m \) is a D3-directing word of \( A \) which is a contradiction. Consequently, the sets \( S_0, S_1, \ldots, S_m \) must be pairwise different. Since \( a_n \in S_k \), \( k = 0, \ldots, m \), the number of these sets can not exceed \( 2^n - 1 \). Therefore, \( |u| \leq 2^{n-1} - 1 \). This ends the proof of Theorem 2.

**Remark 1.** It is worth noting that the proof above with a small changing can be applied for the general case, and one obtains the upper bound \( 2^n - 1 \) for \( d_3(n) \) which is a significant improvement of the upper bound, given in [8].

Now, we study D1-directable trapped c.n.d. automata. By a slight modification of the automata, introduced by Burkhard [1], we prove the following sharp bound.

**Theorem 3.** For any \( n \geq 1 \), \( c_d^T(n) = 2^{n-1} - 1 \).

**Proof.** First we prove that \( 2^{n-1} - 1 \) is a lower bound for \( c_d^T \). To do so, for every integer \( n > 1 \), we present a D1-directable trapped c.n.d. automaton, having a minimum-length D1-directing word \( w \) with \( |w| = 2^{n-1} - 1 \).

Let us define the c.n.d. automaton \( A_n = ([n], X^{(n)}) \) as follows. For every integer \( 2 \leq k \leq n - 2 \), let us consider all of the \( k \)-element subsets of the set \( A' = \{2, \ldots, n\} \). Let us order these sets in a chain such that the first set is \( \{n - k, \ldots, n - 1\} \) and the last one is \( \{n - k + 1, \ldots, n\} \). We denote this sequence by \( A_1^{(k)}, \ldots, A_{(n-k)}^{(k)} \). Now, let \( X_k = \{x_r^{(k)} : r = 1, \ldots, (n-1)_k\} \), \( V = \{v_1, \ldots, v_{n-1}\} \), \( Y = \{y_1, \ldots, y_{n-2}\} \), and

\[
X^{(n)} = V \cup Y \cup (\bigcup \{X_k : 2 \leq k \leq n - 2\}).
\]

The transitions of \( A_n \) are defined as follows. For any \( x \in X^{(n)} \), let \( 1x = \{1\} \). Moreover, for any \( x_r^{(k)} \in X_k \), \( v_l \in V \), \( y_s \in Y \), and state \( j \in A' \), let

\[
jv_t = \begin{cases} 
\{j - 1\} & \text{if } t = j - 1, \\
A' & \text{otherwise},
\end{cases}
\]

\[
jx_r^{(k)} = \begin{cases} 
A_r^{(k)} & \text{if } j \in A_r^{(k)}, \\
A_{r+1}' & \text{otherwise},
\end{cases}
\]

and \( jz = \emptyset \), for all \( j = 1, \ldots, n - 1 \). Now, redefine the remaining transitions as follows. If \( ax = b \), then let \( ax = \{b\} \) be the new transition. Then we obtain an n.d. automaton of \( n \) states whose shortest D3-directing words are of length \((n - 2)^2 + 1\).
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\[ Jy_s = \begin{cases} 
A^{(s)} & \text{if } 2 \leq s \leq n - 2 \land n - s \leq j \leq n, \\
\{n\} & \text{if } s = 1 \land j \in \{n - 1, n\}, \\
A' & \text{otherwise.}
\end{cases} \]

Obviously, \( A_n \) is a trapped c.n.d. automaton, its trap is the state 1. Let us consider the word \( w \in X^{(n)*} \), given by

\[ w = y_{n-2}x_1^{(n-2)} \cdots x_{(n-2)}^{(n-3)} \cdots y_{n-3} \cdots y_2x_{1,2}^{(2)} \cdots x_{(n-1)}^{(2)} \cdots y_1v_{n-1} \cdots v_1. \]

It is easy to check that \( w \) is a D1-directing word of \( A_n \), namely \([n]w = \{1\}\). Moreover, \( w \) is the unique minimum-length D1-directing word of \( A_n \). This fact is based on the following observation.

If \( px \) is a prefix of \( w \), then for any \( x' \in X^{(n)} \), different from \( x \), there exists a prefix \( q \) of \( p \) such that \([n]px' = [n]q\).

Since \( w \) is a minimum-length D1-directing word of \( A_n \) and its length is equal to \( 2^{n-1} - 1 \), we obtain \( 2^{n-1} - 1 \leq cd^1_T(n) \).

Regarding the upper bound, let us observe that if \( w = x_1 \ldots x_m \) is a minimum-length D1-directing word of a trapped c.n.d. automaton \( A = (A, X) \) of \( n(> 1) \) states with a trap \( a_n \in A \), then \( Aw = \{a_n\} \). Moreover, the sequence \( A, Ax_1, \ldots, Ax_1 \ldots x_m \) consists of pairwise different nonempty subsets of \( A \) and each of them contains \( a_n \). The number of these subsets is at most \( 2^{n-1} \), and so, the length of \( w \) is not greater than \( 2^{n-1} - 1 \). Hence, we obtain that \( cd^1_T(n) \leq 2^{n-1} - 1 \).

The statement is obviously also valid for \( n = 1 \). This ends the proof of Theorem 3.

In what follows, we shall use the following observation.

**Lemma.** For every \( n \geq 1 \), \( cd^1_T(n) = cd^2_T(n) \) and \( d^1_T(n) = d^2_T(n) \).

**Proof.** Let us observe that for any trapped n.d. automaton \( A = (A, X) \) of \( n \) states, \( D_1(A) = D_2(A) \). Indeed, \( D_1(A) \subseteq D_2(A) \) follows from the definition. Now, let \( w \in D_2(A) \). Then \( aw = bw \) for every pair of states. This yields that \( \{a_n\} = a_nw = aw \) is valid for any state \( a \in A \), where \( a_n \) denotes the trap state of \( A \). This means that \( w \in D_1(A) \), implying \( D_2(A) \subseteq D_1(A) \). Therefore, \( D_1(A) = D_2(A) \). From this equality it follows that \( cd^1_T(n) = cd^2_T(n) \) and \( d^1_T(n) = d^2_T(n) \).

Now, we can conclude the following statement from Theorem 3 by our Lemma.

**Theorem 4.** For any \( n \geq 1 \), \( cd^2_T(n) = 2^{n-1} - 1 \).

For D1- and D2-directable trapped n.d. automata, we have the following bounds.

**Theorem 5.** For any \( n \geq 1 \), \( 2^{n-1} - 1 \leq d^1_T(n) = d^2_T(n) \leq 2(2^{n-1} - 1) \).
Proof. \( d_T^1(n) = d_T^2(n) \) is provided by our Lemma. By Theorem 3, we have that \( 2^{n-1} - 1 \leq c_2^1(n) \). On the other hand, \( c_T^1(n) \leq d_T^1(n) \), and therefore, \( 2^{n-1} - 1 \leq d_T^1(n) \).

Regarding the upper bound, let us consider an arbitrary D1-directable trapped n.d. automaton \( A = \{\{a_1, \ldots, a_n\}, X\} \) with \( n \geq 2 \). Without loss of generality, we may suppose that \( a_n \) is the trap of \( A \). First, let us observe that \( A \) is a D3-directable n.d. automaton, as well. Let \( w_1 \) be a minimum-length D3-directing word of \( A \). By Theorem 2, \( |w_1| \leq 2^{n-1} - 1 \). Since \( A \) is a trapped n.d. automaton, \( a_n \in a_jw_1 \), for all \( j \in [n] \). Then for every \( j \in [n] \) and \( p \in X^* \), \( a_jw_1p \neq \emptyset \). Now, let \( w_2 = x_1 \ldots x_m \) be a minimum-length word such that \( Aw_2 = \{a_n\} \). Such a word there exists since \( A \) is D1-directable. Let us consider the sequence \( A, Ax_1, \ldots, Ax_1 \ldots x_m \). We show that these sets are pairwise different. If it is not so, then there are integers \( 0 < r < s < m \) such that \( Ax_1 \ldots x_r = Ax_1 \ldots x_s \). Then \( Ax_1 \ldots x_r x_{s+1} \ldots x_m = \{a_n\} \) which is a contradiction. Since \( a_n \in Ap \) for every prefix \( p \) of \( w_2 \), we obtain that \( m \leq 2^{n-1} - 1 \). Now, we prove that \( w_1w_2 \) is a D1-directing word of \( A \). Let \( j \in [n] \) be arbitrary. Then \( a_n \in a_jw_1 \) and \( a_jw_1 \subseteq A \). Moreover, \( a_jw_1w_2 \neq \emptyset \) and \( a_jw_1w_2 \subseteq Aw_2 = \{a_n\} \), and hence, \( a_jw_1w_2 = \{a_n\} \). On the other hand, \( |w_1w_2| = 2^{n-1} - 1 + 2^{n-1} - 1 = 2(2^{n-1} - 1) \). Consequently, \( d_T^1(n) \leq 2(2^{n-1} - 1) \) if \( n \geq 2 \). On the other hand, \( d_T^1(n) \leq 0 \) is obvious. This completes the proof of Theorem 5. \( \square \)

Remark 2. Since \( cd_T^2(n) \leq cd_2(n) \leq d_2(n) \), we obtain that \( 2^{n-1} - 1 \) is a lower bound for both \( cd_2(n) \) and \( d_2(n) \). On the other hand, the known lower bound, given for \( cd_2(n) \) and \( d_2(n) \) in [8], is \( [n^{3/2} - 1]! \) which is less than \( 2^{n-1} - 1 \). Therefore, \( 2^{n-1} - 1 \) is an improvement of the lower bounds of both \( cd_2(n) \) and \( d_2(n) \).

Remark 3. The upper bound, presented in Theorem 5, is worse than the upper bound \( 2^n - n - 1 \), given for \( d_1(n) \) in [8]. The verification of the inequality \( d_1(n) \leq 2^n - n - 1 \) is based on the observation that if \( w = x_1 \ldots x_m \) is a minimum-length D1-directing word of an n.d. automaton \( A = (A, X) \), then the sets \( A, Ax_1, \ldots, Ax_1 \ldots x_m \) must be pairwise different. The following example shows that this observation is not valid, moreover, \( 2^n - n - 1 \) is not necessarily upper bound for \( d_1(n) \) in general. Let \( A = \{(0, 1), \{x, y\}\} \), where \( 0x = \{0, 1\} \), \( 1x = \{1\} \), \( 0y = \emptyset \), and \( 1y = 1 \). Then \( xy \) is a minimum-length D1-directing word of \( A \), but \( \{0, 1\}x = \{0, 1\} \). Moreover, \( 2 = |xy| \leq 2^2 - 2 - 1 \).

References


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