Acta Cybernetica **17** (2006) 459–470.

Finitely Presentable Tree Series

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Abstract

Tree height is known to be a non-recognizable series. In this paper, we detect two remarkable classes where this series belongs: that of polynomially presentable tree series and that of almost linearly presentable tree series.

Both the above classes have nice closure properties, and seem to constitute the first levels of a tree series hierarchy which starts from the class of recognizable treeseries.

1 Introduction

It is well known that some tree functions of wide use in computer science fail to be recognizable, that is they can not be obtained as behaviors of tree automata weighted over a certain semiring. Berstel and Reutenauer proved that the tree series *height* : $T_\Gamma \to \mathbb{N}$ sending every tree t over the ranked alphabet Γ to its height is non-recognizable (cf. [BR]). Therefore it is quite natural to search for classes having good closure properties in which this tree series belongs.

In this paper, we give two such classes: the class PP of polynomially presentable tree series and the class ALP of almost linearly presentable tree series.

Both PP and ALP are closed under sum, scalar product, top-catenation, left derivative and semiring morphism.

Given a finite ranked alphabet Γ and a semiring K we denote by $K\langle\langle T_\Gamma\rangle\rangle$ the set of all tree series $S : T_\Gamma \to K$, equipped with the standard operations of sum, scalar product and top-catenation.

We say that a tree series $S : T_\Gamma \to K$ is *polynomially presentable* whenever it belongs to a finitely generated invariant subalgebra of $K\langle\langle T_\Gamma\rangle\rangle$. Also, $S : T_\Gamma \to K$ is said to be *linearly presentable* whenever it belongs to a finitely generated invariant K-subsemimodule of $K\langle\langle T_\Gamma\rangle\rangle$.

The reader is assumed to be familiar with semirings, semimodules etc (for details, see [SS], [KS]).

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2 Basic Facts

2.1 Trees

In this subsection we briefly exhibit the tree substitution operations used throughout this paper.

Given a finite ranked alphabet $\Gamma = (\Gamma_k)_{k\geq 0}$ and a set of variables $X_n = \{x_1, \ldots, x_n\}$, we denote by $T_\Gamma(X_n)$ the smallest set verifying next two items:

- $\Gamma_0 \cup X_n \subseteq T_\Gamma(X_n)$ and
- for $f \in \Gamma_k$, $k \geq 1$, and $t_1, \ldots, t_k \in T_\Gamma(X_n)$ the word $f(t_1, \ldots, t_k) \in T_\Gamma(X_n)$.

For $n = 0$, $T_\Gamma(X_n)$ is written as T_Γ. The elements of $T_\Gamma(X_n)$ are called trees over Γ indexed by the variables $x_1, \ldots, x_n$.

The *height* of a tree $t \in T_\Gamma$, denoted by *height* (t) is inductively defined by

- $height(c) = 0$, for all $c \in \Gamma_0$ and
- $height(f(t_1, \ldots, t_n)) = 1 + \max\{height(t_i) \mid 1 \leq i \leq n\}$.

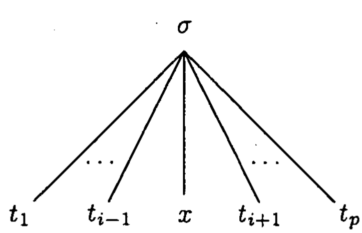
Consider trees

$$t \in T_\Gamma(X_n), t_1, \ldots, t_n, t_1^{(i)}, \ldots, t_{\lambda_i}^{(i)} \in T_\Gamma(X_n), 1 \leq i \leq n$$

where we assume that the variable x_i occurs exactly $\lambda_i \geq 0$ times in the tree t. We use the notation:

- $t[t_1/x_1, \ldots, t_n/x_n]$ or simply $t[t_1, \ldots, t_n]$ for the result of substituting t_i for every occurrence of x_i in t.
- $t\left[\left(t_1^{(1)}, \ldots, t_{\lambda_1}^{(1)}\right)/x_1, \ldots, \left(t_1^{(n)}, \ldots, t_{\lambda_n}^{(n)}\right)/x_n\right]$ for the result of substituting $t_1^{(i)}, \ldots, t_{\lambda_i}^{(i)}$ for the occurrences of x_i in t from left to right $(1 \leq i \leq n)$.

Consider now the subset P_Γ of $T_\Gamma(x)$ consisting of all trees where the variable x occurs once. P_Γ becomes a monoid, with multiplication the substitution at x; precisely, if τ, $\pi \in P_\Gamma$, $\tau\pi$ is the tree obtained by substituting π for x in τ. Actually, P_Γ is the free monoid generated by the trees of the following form:



$\sigma \in \Gamma_p$, $p \geq 1$, $t_j \in T_\Gamma$, $(j \neq i)$.

Figure 1:

On other hand, P_Γ acts canonically on T_Γ :

$$P_\Gamma \times T_\Gamma \to T_\Gamma \qquad (\tau, t) \mapsto \tau t = \tau\,[t/x]\,.$$

For $\tau \in P_\Gamma$, $|\tau|$ denotes its length in the free monoid P_Γ. If τ is as in Figure 1 then $|\tau| = 1$ while if $\tau = \tau_1 \cdot \tau_2$, then $|\tau| = |\tau_1| + |\tau_2|$.

2.2 Formal Series on Trees

Assume a ranked alphabet Γ and a set of variables $X_n = \{x_1, \ldots, x_n\}$ are given, as well as a semiring K.

The functions $S : T_\Gamma(X_n) \to K$ are called *tree series.*

The value of S at $t \in T_\Gamma(X_n)$ is denoted by (S, t) and is refered to as the *coefficient* of S at t.

The set $K\langle\langle T_\Gamma(X_n)\rangle\rangle$ of tree series on $T_\Gamma(X_n)$ is converted into a K-semimodule when addition and scalar multiplication are point wisely defined:

$$\begin{aligned}(S_1 + S_2, t) &= (S_1, t) + (S_2, t)\\ (\lambda S, t) &= \lambda\,(S, t)\end{aligned}$$

for all $t \in T_\Gamma(X_n)\,, \lambda \in K$ and $S_1, S_2, S \in K\langle\langle T_\Gamma(X_n)\rangle\rangle$.

Moreover a partial infinite addition on $K\langle\langle T_\Gamma(X_n)\rangle\rangle$ can be defined as follows: we say that a family of tree series $(S_i)_{i\in I}$ is *locally finite* whenever for each $t \in T_\Gamma(X_n)$ the set $\{i \mid (S_i, t) \neq 0\}$ is finite. Then $\sum\limits_{i\in I} S_i$ exists and is given by

$$\left(\sum_{i\in I} S_i, t\right) = \sum_{i\in I}(S_i, t) \qquad \text{for all } t \in T_\Gamma(X_n)\,.$$

According to this discussion every $S \in K\langle\langle T_\Gamma(X_n)\rangle\rangle$ can be represented as an infinite sum

$$S = \sum_{t\in T_\Gamma(X_n)} (S, t)\, t.$$

The support of a series $S : T_\Gamma(X_n) \to K$ is the tree language

$$supp\,(S) = \{t \in T_\Gamma(X_n) \mid (S, t) \neq 0\}\,.$$

Series $S \in K\langle\langle T_\Gamma(X_n)\rangle\rangle$ whose support is finite are termed *polynomials* and their set is denoted by $K\langle T_\Gamma(X_n)\rangle$.

Given $\sigma \in \Gamma_p$ and $S_1, \ldots, S_p \in K\langle\langle T_\Gamma(X_n)\rangle\rangle$, the *$\sigma$-top catenation series*

$$\sigma\,(S_1, \ldots, S_p) : T_\Gamma(X_n) \to K$$

is defined as follows. For $t \in T_\Gamma(X_n)$

$$(\sigma\,(S_1, \ldots, S_p)\,, t) = (S_1, t_1) \cdots (S_p, t_p) \text{ if } t = \sigma\,(t_1, \ldots, t_p) \text{ and } 0 \text{ else.}$$

More generally, for every $n \geq 0$, $t \in T_\Gamma(X_n)$ and $S_1, \ldots, S_n \in K\langle\langle T_\Gamma\rangle\rangle$ we define the series

$$t[S_1, \ldots, S_n] : T_\Gamma(X_n) \to K$$

inductively by the clauses

- $x_i[S_1, \ldots, S_n] = S_i, \qquad 1 \leq i \leq n$
- $c[S_1, \ldots, S_n] = c, \qquad c \in \Gamma_0$
- $\sigma(t_1, \ldots, t_p)[S_1, \ldots, S_n] = \sigma(t_1[S_1, \ldots, S_n], \ldots, t_p[S_1, \ldots, S_n])$, for $\sigma \in \Gamma_p, t_j \in T_\Gamma(X_n)$.

Proposition 1. *For every $n \geq 1, t \in T_\Gamma(X_n), S_1, \ldots, S_n \in K\langle\langle T_\Gamma\rangle\rangle$ and $s \in T_\Gamma$,*

$$(t[S_1, \ldots, S_n], s) = \left(S_1, t_1^{(1)}\right) \cdots \left(S_1, t_{\lambda_1}^{(1)}\right) \cdots \left(S_n, t_1^{(n)}\right) \cdots \left(S_n, t_{\lambda_n}^{(n)}\right)$$

if there are $t_1^{(i)}, \ldots, t_{\lambda_i}^{(i)} \in T_\Gamma$, $1 \leq i \leq n$ such that

$$s = t\left[\left(t_1^{(1)}, \ldots, t_{\lambda_1}^{(1)}\right)/x_1, \ldots, \left(t_1^{(n)}, \ldots, t_{\lambda_n}^{(n)}\right)/x_n\right].$$

and $(t[S_1 \ldots, S_n], s) = 0$, otherewise.

By linear extension, we can define $p[S_1, \ldots, S_n]$ for any polynomial $p \in K\langle T_\Gamma(X_n)\rangle$

$$p[S_1, \ldots, S_n] = \sum_{t \in T_\Gamma(X_n)} (p, t)\, t[S_1, \ldots, S_n].$$

The last operation we need is derivation. The derivative of $S \in K\langle\langle T_\Gamma\rangle\rangle$ at $\tau \in P_\Gamma$ is a tree series

$$\tau^{-1}S = \sum_{t \in T_\Gamma} (S, \tau t)\, t.$$

The derivation has the following properties:

1. $\tau^{-1}\left(\pi^{-1}S\right) = (\pi\tau)^{-1} S$, for all $\tau, \pi \in P_\Gamma, S \in K\langle\langle T_\Gamma\rangle\rangle$,
2. $\tau^{-1}(\sigma(S_1, \ldots, S_p)) = \prod_{j \neq i} (S_j, t_j)\, \pi^{-1}S_i$, if $\tau = \sigma(t_1, \ldots, t_{i-1}, \pi, t_{i+1}, \ldots, t_p)$, for every $\tau \in P_\Gamma$ and $S_1, \ldots, S_p \in K\langle\langle T_\Gamma\rangle\rangle$,
3. for every $\tau \in P_\Gamma$, index set I and family $(S_i, i \in I)$ over $K\langle\langle T_\Gamma\rangle\rangle$, if $\sum_{i \in I} S_i$ exists, then $\sum_{i \in I} \tau^{-1}S_i$ also exists and $\tau^{-1}\left(\sum_{i \in I} S_i\right) = \sum_{i \in I} \tau^{-1}S_i$.

2.3 Recognizable Tree series

Recall that a K-Γ-tree automaton is a triple $\mathcal{M} = (Q, \mu, T)$ consisting of a finite set Q of states, a final state function $T : Q \to K$ and a Γ-indexed family of functions

$$\mu = \left(\mu_f : Q^n \to K^Q\right)_{f \in \Gamma_n, n \geq 0}$$

describing the moves of $\mathcal{M}$.

The function $\mu_f : Q^n \to K^Q$ is multilinearly extended into a function

$$\bar{\mu}_f : \left(K^Q\right)^n \to K^Q$$

by the formula

$$\bar{\mu}_f (X_1, \ldots, X_n) = \sum_{q_1, \ldots, q_n \in Q} X_1 (q_1) \cdots X_n (q_n) \, \mu_f (q_1, \ldots, q_n) .$$

Then the behaviour of $\mathcal{M}$ is the series $|\mathcal{M}| : T_\Gamma \to K$ defined by

$$(|\mathcal{M}|, t) = \sum_{q \in Q} \mu_{\mathcal{M}} (t) (q) \cdot T (q)$$

where $\mu_{\mathcal{M}} : T_\Gamma \to K^Q$ is inductively given by the clause

$$\mu_{\mathcal{M}} (f (t_1, \ldots, t_n)) = \bar{\mu}_f (\mu_{\mathcal{M}} (t_1), \ldots, \mu_{\mathcal{M}} (t_n)), \; f \in \Gamma_n, n \geq 0, t_1, \ldots, t_n \in T_\Gamma .$$

A tree series $S : T_\Gamma \to K$ is called *recognizable* whenever it is the behaviour of a K-Γ-tree automaton. $REC\,(K, \Gamma)$ stands for the so obtained class.

The tree series $S : T_\Gamma \to \mathbb{N}$ sending every tree $t \in T_\Gamma$ to its size (i.e. the number of symbols of Γ occurring in t), is recognizable. On the contrary, the tree series $height : T_\Gamma \to \mathbb{N}$ fails to be recognizable (cf. [BR]).

3 Subalgebras of $K \langle\langle T_\Gamma \rangle\rangle$

A subset $\mathcal{A} \subseteq K \langle\langle T_\Gamma \rangle\rangle$ closed under sum, scalar product and σ-top catenation (for all $\sigma \in \Gamma_p, p \geq 1$) is termed a subalgebra of $K \langle\langle T_\Gamma \rangle\rangle$.

Proposition 2. *$\mathcal{A} \subseteq K \langle\langle T_\Gamma \rangle\rangle$ is a subalgebra iff for each polynomial $p \in K \langle T_\Gamma (X_n) \rangle$ and a sequence of series $S_1, \ldots, S_n \in \mathcal{A}, p [S_1, \ldots, S_n] \in \mathcal{A}$.*

The intersection of any family of subalgebras of $K \langle\langle T_\Gamma \rangle\rangle$ is again a subalgebra and thus we can speak of the subalgebra generated by a subset $\mathcal{S} \subseteq K \langle\langle T_\Gamma \rangle\rangle$. It is denoted by $\langle \mathcal{S} \rangle_{K,\Gamma}$.

Proposition 3. *For every $\mathcal{S} \subseteq K \langle\langle T_\Gamma \rangle\rangle$, we have*
$\langle \mathcal{S} \rangle_{K,\Gamma} = \{ p [S_1, \ldots, S_n] \mid p \in K \langle T_\Gamma (X_n) \rangle, S_1, \ldots, S_n \in \mathcal{S}, n \geq 0 \}.$

Proof. Let

$$U = \{p[S_1, \dots, S_n] \mid p \in K\langle T_\Gamma(X_n)\rangle, S_1, \dots, S_n \in \mathcal{S}, n \geq 0\}.$$

Certainly $\mathcal{S} \subseteq U$. Next we show that U is a subalgebra of $K\langle\langle T_\Gamma\rangle\rangle$, i.e., that for every $n \geq 0, p \in K\langle T_\Gamma(X_n)\rangle$ and $p_i \in K\langle T_\Gamma(X_{k_i})\rangle, S_1^{(i)}, \dots, S_{k_i}^{(i)} \in \mathcal{S}$ $1 \leq i \leq n$ it holds

$$p\left[p_1\left[S_1^{(1)}, \dots, S_{k_1}^{(1)}\right], \dots, p_n\left[S_1^{(n)}, \dots, S_{k_n}^{(n)}\right]\right] \in U.$$

We introduce the polynomial $\bar{p}_i, 1 \leq i \leq n$ by setting

$$\bar{p}_1 = p_1,$$

$$\bar{p}_2 = p_2\left[x_{k_1+1}/x_1, \dots, x_{k_1+k_2}/x_{k_2}\right],$$

$$\vdots$$

$$\bar{p}_n = p_n\left[x_{k_1+\dots+k_{n-1}+1}/x_1, \dots, x_{k_1+\dots+k_{n-1}+k_n}/x_{k_n}\right].$$

Then

$$p\left[p_1\left[S_1^{(1)}, \dots, S_{k_1}^{(1)}\right], \dots, p_n\left[S_1^{(n)}, \dots, S_{k_n}^{(n)}\right]\right] =$$

$$p[\bar{p}_1, \dots, \bar{p}_n]\left[S_1^{(1)}/x_1, \dots, S_{k_1}^{(1)}/x_{k_1}, S_1^{(2)}/x_{k_1+1}, \dots,\right.$$

$$\left.S_{k_2}^{(2)}/x_{k_1+k_2}, \dots, S_1^{(n)}/x_{k_1+\dots+k_{n-1}+1}, \dots, S_{k_n}^{(n)}/x_{k_1+\dots+k_n}\right].$$

Since $p[\bar{p}_1, \dots, \bar{p}_n] \in K\langle\langle T_\Gamma(X_{k_1+\dots+k_n})\rangle\rangle$ the result comes by aplying Proposition 2.

Now, let $\overline{U}$ be a subalgebra of $K\langle\langle T_\Gamma\rangle\rangle$ including $\mathcal{S}$. Then for any $p[S_1, \dots, S_n] \in U$ with $p \in K\langle T_\Gamma(X_n)\rangle$ and $S_1, \dots, S_n \in \mathcal{S}$, we have $p[S_1, \dots, S_n] \in \overline{U}$ and thus U is the smallest subalgebra of $K\langle\langle T_\Gamma\rangle\rangle$ including $\mathcal{S}$, i.e. $U = \langle\mathcal{S}\rangle_{K,\Gamma}$. ☐

A subalgebra $\mathcal{A} \subseteq K\langle\langle T_\Gamma\rangle\rangle$ is said to be *invariant* if it is closed under derivation, i.e.

$$S \in \mathcal{A} \text{ and } \tau \in P_\Gamma \text{ implies } \tau^{-1}S \in \mathcal{A}.$$

Proposition 4. *The subalgebra $\langle\mathcal{S}\rangle$ generated by $\mathcal{S} \subseteq K\langle T_\Gamma\rangle$ is invariant iff it contains the derivatives of all its generators*

Proof. One direction is obvious.

To establish the opposite direction we first show that if $\sigma \in \Gamma_k$ and $S_1, \dots, S_k \in \mathcal{S}$ then

$$\tau^{-1}\sigma(S_1, \dots, S_k) \in \langle\mathcal{S}\rangle, \quad \text{for all } \tau \in P_\Gamma.$$

Indeed, if $\tau = \sigma(t_1, \dots, t_{i-1}, \pi, t_{i+1}, \dots, t_k)$ then for all $t \in T_\Gamma$

$$\begin{aligned}(\tau^{-1}\sigma(S_1, \dots, S_k), t) &= (\sigma(S_1, \dots, S_k), \tau t) \\ &= \prod_{j \neq i}(S_j, t_j)(S_i, \pi t) \\ &= \alpha(\pi^{-1}S_i, t),\end{aligned}$$

where $\alpha = \prod_{j \neq i} (S_j, t_j)$. In other words

$$\tau^{-1}\sigma(S_1, \ldots, S_k) = \alpha\pi^{-1}S_i \in \langle \mathcal{S} \rangle, \quad \alpha \in K$$

since, by hypothesis, $\langle \mathcal{S} \rangle$ contains all the derivatives of its generators.

In all other instances of τ, it holds

$$\tau^{-1}\sigma(S_1, \ldots, S_k) = 0 \in \langle \mathcal{S} \rangle.$$

By induction on the complexity of $t \in T_\Gamma(X_n)$ we show that $\tau^{-1}t[S_1, \ldots, S_n] \in \langle \mathcal{S} \rangle$.

For $t \in \Gamma_0 \cup X_n$ we have nothing to prove. Let $t = \sigma(t_1, \ldots, t_k)$; then

$$\begin{aligned} \tau^{-1}t[S_1, \ldots, S_n] &= \tau^{-1}\sigma(t_1, \ldots, t_k)[S_1, \ldots, S_n] \\ &= \tau^{-1}\sigma(t_1[S_1, \ldots, S_n], \ldots, t_k[S_1, \ldots, S_n]) \in \langle \mathcal{S} \rangle \end{aligned}$$

Furthermore, for any polynomial $p \in K\langle T_\Gamma(X_n) \rangle$ we have

$$\tau^{-1}p[S_1, \ldots, S_n] = \Sigma_{t \in T_\Gamma(X_n)}(p, t)\tau^{-1}t[S_1, \ldots, S_n] \in \langle \mathcal{S} \rangle$$

where the above sum is finite.

□

4 Finitely Presentable Tree series

A series $S \in K\langle\langle T_\Gamma \rangle\rangle$ is said to be *linearly presentable* if there exist series $S_1, \ldots, S_n \in K\langle\langle T_\Gamma \rangle\rangle$ with the following two properties

1. There are $\lambda_1, \ldots, \lambda_n \in K$ such that S is expressed as a linear combination of them
$$S = \lambda_1 S_1 + \cdots + \lambda_n S_n, \ \lambda_i \in K$$
and
2. for each index i $(1 \leq i \leq k)$ and each $\tau \in P_\Gamma$, there are $\mu_{i_1}, \ldots, \mu_{i_k} \in K$ such that for each index i $(1 \leq i \leq n)$ and each $\tau \in P_\Gamma$
$$\tau^{-1}S_i = \sum_{i=1}^{n} \mu_{ij} S_j, \ \mu_{ij} \in K, 1 \leq i \leq n.$$

We denote by $LP(K, \Gamma)$ the class of linearly presentable tree series.

Proposition 5. $REC(\Gamma, K) \subseteq LP(\Gamma, K)$.

Proof. Consider a K-Γ-tree automaton $\mathcal{M} = (Q, \mu, T)$, its associated system

$$x_q = \sum_{\substack{k \geq 0, f \in \Gamma_k \\ q_1, \ldots, q_k \in Q}} \mu_f(q_1, \ldots, q_k)(q) f(x_{q_1}, \ldots, x_{q_k}) \qquad (\Sigma M)$$

and for all $q \in Q$, the K-Γ-tree automaton $\mathcal{M}_q = (Q, \mu, \hat{q})$ with $\hat{q} : Q \to K$ defined by $\hat{q}(p) = 1$, if $p = q$ and $\hat{q}(p) = 0$, else.

It is known [Bo2] that the tuple $(|\mathcal{M}_q|)_{q \in Q}$ is the unique solution of (ΣM)

$$|\mathcal{M}_q| = \sum_{\substack{f \in \Gamma_k, k \geq 0 \\ q_1, \ldots, q_k \in Q}} \mu_f (q_1, \ldots, q_k)(q)\, f(|\mathcal{M}_{q_1}|, \ldots, |\mathcal{M}_{q_k}|). \qquad (\star)$$

By construction we have

$$|\mathcal{M}| = \sum_{q \in Q} T(q)\, |\mathcal{M}_q|$$

that is $|\mathcal{M}|$ is linear combination of the series $|\mathcal{M}_q|, q \in Q$. The proof will be completed if for each tree $\tau \in P_\Gamma$ of the form $\tau = g(t_1, \ldots, t_{i-1}, x, t_{i+1}, \ldots, t_k), g \in \Gamma_k, t \in T_\Gamma$ and for each state $q \in Q$ show that $\tau^{-1} |\mathcal{M}_q|$ can also be written as linear combination of $|\mathcal{M}_q|, q \in Q$.

Derivating $(\star)$ at τ we get

$$\begin{aligned}
\left(\tau^{-1} |\mathcal{M}_q|, s\right) &= (|\mathcal{M}_q|, \tau s) \\
&= \sum_{\substack{f \in \Gamma_k, k \geq 0 \\ q_1, \ldots, q_k \in Q}} \mu_f (q_1, \ldots, q_k)(q) \left(f(|\mathcal{M}_{q_1}|, \ldots, |\mathcal{M}_{q_k}|), \tau s\right) \\
&= \sum_{q_1, \ldots, q_k \in Q} \mu_g (q_1, \ldots, q_k)(q) \left(g(|\mathcal{M}_{q_1}|, \ldots, |\mathcal{M}_{q_k}|), \tau s\right) \\
&= \sum_{q_1, \ldots, q_k \in Q} \mu_g (q_1, \ldots, q_k)(q) (|\mathcal{M}_{q_1}|, t_1) \cdots \left(|\mathcal{M}_{q_{i-1}}|, t_{i-1}\right) \\
&\qquad (|\mathcal{M}_{q_i}|, s) \left(|\mathcal{M}_{q_{i+1}}|, t_{i+1}\right) \cdots (|\mathcal{M}_{q_k}|, t_k) \\
&= \sum_{q_i \in Q} \lambda_{q_i, \tau} (|\mathcal{M}_{q_i}|, s).
\end{aligned}$$

In other words

$$\tau^{-1} |\mathcal{M}_q| = \sum_{q_i \in Q} \lambda_{q_i, \tau} |\mathcal{M}_{q_i}|$$

as wanted. □

A tree series $S \in K \langle\langle T_\Gamma \rangle\rangle$ is said to be *polynomially presentable* if there is a finite subset $\mathcal{S} \subseteq K \langle\langle T_\Gamma \rangle\rangle$ satisfying the following two conditions:

1. there is a polynomial $p \in K \langle T_\Gamma (X_n) \rangle$ and there are $S_1, \ldots, S_n \in \mathcal{S}$ such that $S = p[S_1, \ldots, S_n]$ and

2. for every $\tau \in P_\Gamma$ and $S \in \mathcal{S}$, there is a polynomial $p_{\tau,S} \in K \langle T_\Gamma (X_n) \rangle$ and $S_1, \ldots, S_n \in \mathcal{S}$ such that $\tau^{-1}(S) = p_{\tau,S} [S_1, \ldots, S_n]$.

Let us denote the class of polynomially presentable tree series by $PP(K,\Gamma)$. It should be clear that linearly presentable tree series are also polynomially presentable, hence $LP(\Gamma,K) \subseteq PP(\Gamma,K)$.

Moreover, by Proposition 2 and the definition of an invariant subalgebra, S is polynomially presentable if and only if it is an element of an invariant, finitely generated subalgebra of $K\langle\langle T_\Gamma\rangle\rangle$.

Closure properties of polynomially presentable tree series are examined below.

Proposition 6. *The family $PP(K,\Gamma)$ of polynomially presentable series of $K\langle\langle T_\Gamma\rangle\rangle$ is an invariant subalgebra of $K\langle\langle T_\Gamma\rangle\rangle$. Moreover if $\phi : K \to \Lambda$ is a semiring morphism and $S \in PP(K,\Gamma)$, then $S\circ\phi \in PP(\Gamma,\Lambda)$, where $(S\circ\phi,t) = \phi(S(t))$, for all $t \in T_\Gamma$.*

Proof. According to Proposition 4, if $\mathcal{A}$ and $\mathcal{A}'$ are invariant subalgebras of $K\langle\langle T_\Gamma\rangle\rangle$ generated by the lists $T_1,\ldots,T_k$ and $T'_1,\ldots,T'_\lambda$ respectively, then the subalgebra generated by the joint list $T_1,\ldots,T_k,T'_1,\ldots T'_\lambda$ is automatically invariant. In other words we may assume that any finite set of finitely presentable series is included into the same invariant finitely generated subalgebra.

Thus, if $S_1,\ldots,S_n \in PP(K,\Gamma)$ and $p \in K\langle T_\Gamma\rangle$, then there exist series $T_1,\ldots,T_k$ so that

$$S_i = p_i[T_1,\ldots,T_k],\ p_i \in K\langle T_\Gamma(X_k)\rangle, i = 1,\ldots,n$$

and for all $\tau \in P_\Gamma$

$$\tau^{-1}T_j = p_{j,\tau}[T_1,\ldots,T_k],\ p_{j,\tau} \in K\langle T_\Gamma(X_k)\rangle, j = 1,\ldots,k.$$

We have

$$p[S_1,\ldots,S_n] = p[p_1[T_1,\ldots,T_k],\ldots,p_n[T_1,\ldots,T_k]] = p[p_1,\ldots,p_n][T_1,\ldots,T_k].$$

Since $p[p_1,\ldots,p_n]$ is polynomial, we get

$$p[S_1,\ldots,S_n] \in PP(K,\Gamma).$$

Therefore, by virtue of Proposition 2, $PP(K,\Gamma)$ is a subalgebra of $K\langle\langle T_\Gamma\rangle\rangle$. Next we establish the following identities

$$\tau^{-1}(\phi\circ S) = \phi\circ\left(\tau^{-1}S\right), \quad \phi\circ(p[S_1,\ldots,S_n]) = (\phi\circ p)[\phi\circ S_1,\ldots,\phi\circ S_n]$$

holding for all $\tau \in P_\Gamma, S, S_1,\ldots,S_n \in K\langle\langle T_\Gamma\rangle\rangle, p \in K\langle T_\Gamma(X_n)\rangle$ and any semiring morphism $\phi : K \to \Lambda$.

Indeed for all $s \in T_\Gamma$ we have

$$\left(\tau^{-1}(\phi\circ S), s\right) = (\phi\circ S, \tau s) = \phi(S,\tau s) = \phi\left(\tau^{-1}S, s\right) = \left(\phi\circ\left(\tau^{-1}S\right), s\right)$$

and

$$\begin{gathered}(\phi \circ (p[S_1, \ldots, S_n]), s) = \phi(p[S_1, \ldots, S_n], s) \\ = \phi(p,t)\left(S_1, s_1^{(1)}\right) \cdots \left(S_1, s_{k_1}^{(1)}\right) \cdots \left(S_1, s_1^{(n)}\right) \cdots \left(S_1, s_{k_n}^{(n)}\right) \\ = (\phi \circ p, t)\left(\phi \circ S_1, s_1^{(1)}\right) \cdots \left(\phi \circ S_1, s_{k_1}^{(1)}\right) \cdots \left(\phi \circ S_1, s_1^{(n)}\right) \cdots \left(\phi \circ S_1, s_{k_n}^{(n)}\right) \\ = ((\phi \circ p)[\phi \circ S_1, \ldots, \phi \circ S_n], s)\end{gathered}$$

where

$$s = t\left[\left(s_1^{(1)}, \ldots, s_{k_1}^{(1)}\right), \ldots, \left(s_1^{(n)}, \ldots, s_{k_n}^{(n)}\right)\right].$$

Now assume that $S \in K\langle\langle T_\Gamma \rangle\rangle$ is polynomially presentable, i.e. there exists a finite list $S_1, \ldots, S_n \in K\langle\langle T_\Gamma \rangle\rangle$ so that

$$S = p[S_{\mu_1}, \ldots, S_{\mu_k}] \text{ and } \tau^{-1} S_i = r_i\left[S_{j_1}, \ldots, S_{j_{\lambda_i}}\right]$$

for some polynomials $p \in K\langle T_\Gamma(X_k)\rangle$, $r_i \in K\langle T_\Gamma(X_{\lambda_i})\rangle$ and $\mu_1, \ldots, \mu_k$, $j_1, \ldots, j_{\lambda_i} \in \{1, 2, \ldots, n\}$, $1 \le i \le n$. Then $\phi \circ S = (\phi \circ p)[\phi \circ S_{\mu_1}, \ldots, \phi \circ S_{\mu_k}]$ and $\tau^{-1}(\phi \circ S_i) = (\phi \circ r_i)\left[\phi \circ S_{j_1}, \ldots, \phi \circ S_{j_{\lambda_i}}\right]$ and so $\phi \circ S$ is again a polynomially presentable series. □

By Proposition 4 and the remark made after the definition of polynomially presentable tree series, we have $REC(K,\Gamma) \subsetneq LP(K,\Gamma) \subsetneq PP(K,\Gamma)$. Next we show that $PP(K,\Gamma) - REC(K,\Gamma) \neq \emptyset$.

Proposition 7. *The series $height : T_\Gamma \to \mathbb{N}$ is polynomially presentable.*

Proof. Let $\mathcal{A}$ be the subalgebra of $\mathbb{N}\langle\langle T_\Gamma \rangle\rangle$ generated by the set $\{\mathbf{1}, height\}$, where $\mathbf{1}$ is the tree series over K and Γ whose all coefficients are equal to 1. Certainly, $height \in \mathcal{A}$. Let us show that $\mathcal{A}$ is invariant. Since, for every $\tau \in P_\Gamma, \tau^{-1}(\mathbf{1}) = \mathbf{1}$, it is sufficient to show that, for every $\tau \in P_\Gamma$, there is a polynomial $p \in \mathbb{N}\langle T_\Gamma(X_2)\rangle$ such that $\tau^{-1}(height) = p[\mathbf{1}, height]$.

We distinct the cases:

Case 1 $height(\tau) \ge |\tau|$. Then

$$\tau^{-1} height = \sum_{k=1}^{n} (height(\tau) - |\tau| - height(t_k))\, t_k + |\tau| \cdot \mathbf{1} + height$$

where $t_1, \ldots, t_n$ are all the trees verifying

$$height(t_k) \le height(\tau) - |\tau|.$$

Case 2 $height(\tau) < |\tau|$. Then it holds

$$\tau^{-1} height = |\tau| \cdot \mathbf{1} + height.$$

Hence, in any case $\tau^{-1} height \in \mathcal{A}$, as claimed. □

Corollary 8. $PP(\mathbb{N},\Gamma) - REC(\mathbb{N},\Gamma) \neq \emptyset$.

Proof. We only have to combine the previous result together with the fact that height is a non-recognizable tree series. $\square$

In [Bol], linearly presentable series are obtained as matrix representations and as behaviours of the so called tree modules. On the other hand, when K is a field, recognizable and linearly presentable series coincide (cf. [BA]).

It is an open question whether $LP(K,\Gamma) - REC(K,\Gamma) \neq \emptyset$ or $PP(K,\Gamma) - LP(K,\Gamma) \neq \emptyset$ for semirings which are not fields.

5 Almost Presentable Tree series

We define the tree series $S, S' : T_\Gamma \to K$ to be *almost equal* and write $S \equiv S'$ whenever $(S,t) = (S',t)$ for all but a finite number of t's.

The above equivalence relation is compatible with sum, scalar product and derivation, i.e.

$$S_i \equiv S_i' \;(i = 1,2), \quad S \equiv S', \quad \lambda \in K, \tau \in P_\Gamma$$

imply

$$S_1 + S_2 \equiv S_1' + S_2', \quad \lambda S \equiv \lambda S', \quad \tau^{-1} S \equiv \tau^{-1} S'.$$

Call a series $S \in K\langle\langle T_\Gamma\rangle\rangle$ *almost linearly* presentable whenever there is a finite list of series $S_1, \ldots, S_n \in K\langle\langle T_\Gamma\rangle\rangle$ such that $S \equiv \lambda_1 S_1 + \cdots + \lambda_n S_n$ for some $\lambda_1, \ldots, \lambda_n \in K$ and for all $\tau \in P_\Gamma$ and $i = 1, \ldots, n$ we have $\tau^{-1} S_i \equiv \mu_1 S_1 + \cdots + \mu_n S_n$ for some $\mu_1, \ldots, \mu_n \in K$.

The tree series *height* is almost linearly presentable since for all $\tau \in P_\Gamma$ it holds

$$\tau^{-1} \mathit{height} \equiv |\tau| \cdot 1 + \mathit{height}.$$

Hence the class $ALP(K,\Gamma)$ *of* almost linearly presentable series properly contains that of almost recognizable tree series.

Moreover $ALP(K,\Gamma)$ is an invariant subalgebra of $K\langle\langle T_\Gamma\rangle\rangle$.

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Received December, 2004