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# Partially Ordered Pattern Algebras

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**Abstract**

A partial order $\preceq$ on a set $A$ induces a partition of each power $A^n$ into "patterns" in a natural way. An operation on $A$ is called a $\preceq$-pattern operation if its restriction to each pattern is a projection. We examine functional completeness of algebras with $\preceq$-pattern fundamental operations.

**Keywords:** majority function, semiprojection, ternary discriminator, dual discriminator, functionally completeness

## 1 Preliminaries

A finite algebra $\mathbf{A} = (A; F)$ is called *functionally complete* if every (finitary) operation on $A$ is a polinomial operation of $\mathbf{A}$. An $n$-ary operation $f$ on $A$ is *conservative* if $f(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\}$ for all $x_1, \ldots, x_n \in A$. An algebra is conservative if all of its fundamental operations are conservative.

A possible approach to the study of conservative operations is to consider them as relational pattern functions or $\rho$-pattern functions. Given a $k$-ary relation $\rho \subseteq A^k$, two $n$-tuples $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n) \in A^n$ are said to be of the same pattern with respect to $\rho$ if for all $i_1, \ldots, i_k \in \{1, \ldots, n\}$ the conditions $(x_{i_1}, \ldots, x_{i_k}) \in \rho$ and $(y_{i_1}, \ldots, y_{i_k}) \in \rho$ mutually imply each other. An operation $f : A^n \to A$ is a *$\rho$-pattern function* if $f(x_1, \ldots, x_n)$ always equals some $x_i$, $i \in \{1, \ldots, n\}$ where $i$ depends only on the $\rho$-pattern of $(x_1, \ldots, x_n)$. In fact, any conservative operation is a $\rho$-pattern function for some $\rho$ — see [11]. An algebra $\mathbf{A}$ is called a *$\rho$-pattern algebra* if its fundamental operations (or equivalently its term operations) are $\rho$-pattern functions for the same relation $\rho$ on $A$. Several facts about functional completeness were proved, for the cases where $\rho$ is an equivalence [9], a central relation [10, 14], a graph of a permutation [13], a bounded partial order [12], or a regular relation [8] on $A$. These relations appear in Rosenberg's primality criterion [6].

In particular if $\preceq$ is a partial order or a linear order on $A$, then a $\preceq$-pattern algebra is called a partially ordered pattern algebra or a linearly ordered pattern algebra. Throughout the paper such algebras will be called *$\preceq$-pattern algebras*.

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The aim of this article is to continue research on functional completeness of finite partially ordered pattern algebras.

In case when the relation $\rho$ on $A$ is the identity the $\rho$-pattern algebra is called *pattern algebra*. The basic operations of pattern algebras are called pattern functions. Pattern functions were first introduced by Quackenbush [5]. B. Csákány [1] proved that every finite pattern algebra $(A; f)$ with $|A| \geq 3$ is functionally complete if $f$ is an arbitrary nontrivial pattern function. The most known examples of pattern algebras are $(A; f)$ and $(A; g)$ where $f$ is the *ternary discriminator* [4] ($f(x, y, z) = z$ if $x = y$ and $f(x, y, z) = x$ if $x \neq y$) and $g$ is the *dual discriminator* [2] ($g(x, y, z) = x$ if $x = y$ and $g(x, y, z) = z$ if $x \neq y$).
We need the following definitions and results.

An $n$-ary relation $\rho$ on $A$ is called *central* iff $\rho \neq A^n$ and

(a) there exists $c \in A$ such that $(a_1, \dots, a_n) \in \rho$ whenever at least one $a_i = c$ (the set of all such $c$'s is called the *center* of $\rho$);

(b) $(a_1, \dots, a_n) \in \rho$ implies that $(a_{1\pi}, \dots, a_{n\pi}) \in \rho$ for every permutation $\pi$ of $\{1, \dots, n\}$ ($\rho$ is *totally symmetric*);

(c) $(a_1, \dots, a_n) \in \rho$ whenever $a_i = a_j$ for some $i \neq j$ ($\rho$ is *totally reflexive*).

Let $A$ be a finite and nonempty set, $k, n \geq 1$, $f$ a $k$-ary function on $A$ and $\rho \subseteq A^n$ an arbitrary $n$-ary relation. The operation $f$ is said to *preserve* $\rho$ if $\rho$ is a subalgebra of the $n$th direct power of the algebra $(A; f)$; in other words, $f$ preserves $\rho$ if for any $k \times n$ matrix $M$ with entries in $A$, whose rows belong to $\rho$, the row obtained by applying $f$ to the columns of $M$ also belongs to $\rho$. Adding this extra row to $M$ we get a so-called $f$-matrix [3].

A ternary operation $f$ on $A$ is a *majority function* if $f(x, x, y) = f(x, y, x) = f(y, x, x) = x$ holds for all $x, y \in A$. An $n$-ary $i$-th *semiprojection* on $A$ $(n \geq 3, 1 \leq i \leq n)$ is an operation $f$ with the property that $f(x_1, x_2, \dots, x_n) = x_i$ whenever at least two of the elements $x_1, \dots, x_n$ are equal. The following proposition was obtained in [13] from Rosenberg's fundamental theorem on minimal clones [7].

**Proposition 1.** *The clone of the term operations of every nontrivial finite $\rho$-pattern algebra* **A** *with at least three elements contains a nontrivial binary $\rho$-pattern function, or a ternary majority $\rho$-pattern function, or a nontrivial $\rho$-pattern function, which is a semiprojection.*

Now we formulate the following theorem (which was got from Proposition 4 in [13]).

**Theorem 2.** *Let* $\mathbf{A} = (A; f)$ *be a finite $\rho$-pattern algebra with* $|A| \geq 3$. *The algebra* $(A; f)$ *is functionally complete iff*

*(a) $f$ is monotonic with respect to no bounded partial order on $A$,*

*(b) $f$ preserves no binary central relations on $A$,*

*(c) $f$ preserves no nontrivial equivalences on $A$.*

# 2 Results

**Theorem 3.** *Let $(A; \preceq)$ be a finite poset with at least three elements that has a least or a greatest element. If $f$ is an arbitrary binary $\preceq$-pattern function on $A$, then the algebra $(A; f)$ is not functionally complete.*

*Proof.* Let $a$ be the least or the greatest element of $(A; \preceq)$. Let $\rho$ be the nontrivial equivalence on $A$ with blocks $\{a\}$, $A \backslash \{a\}$. Now $f$ preserves $\rho$ and apply Theorem 2. □

**Remark.** Let $\underline{n} = \{0, 1, \ldots, n-1\}$ be an at least three-element set, and let $\preceq$ be a linear order on $\underline{n}$ such that $0 \preceq i \preceq n-1$ holds for each $i \in \underline{n}$. If $a, b \in \underline{n}$ and $a \preceq b$ but $a \neq b$ then we write $a \prec b$. Now the following statement is true.

If $\pi$ and $\sigma$ are two different permutations of the set $\{1, 2, \ldots, k\}$ then the $k$-tuples $(a_{1\pi}, a_{2\pi}, \ldots, a_{k\pi})$, $(a_{1\sigma}, a_{2\sigma}, \ldots, a_{k\sigma})$ are not in the same pattern with respect to $\preceq$ where $a_1, a_2, \ldots, a_k \in \underline{n}$ with $a_1 \prec a_2 \prec \ldots \prec a_k$.

Now we can formulate the following theorem.

**Theorem 4.** *Let $(A; \preceq)$ be a finite linearly ordered set with $|A| = n \geq 4$, and let $f$ be a $\preceq$-pattern function that is a majority function on $A$. Then the algebra $(A; f)$ is functionally complete iff for arbitrary elements $a_1, a_2, a_3 \in A$ with $a_1 \prec a_2 \prec a_3$ exactly one of the following statements holds:*

*(a) there exist permutations $\pi, \sigma$ of the set $\{1, 2, 3\}$ for which the values $f(a_1, a_2, a_3)$, $f(a_{1\pi}, a_{2\pi}, a_{3\pi})$, $f(a_{1\sigma}, a_{2\sigma}, a_{3\sigma})$ are pairwise distinct,*

*(b) $f(a_{1\pi}, a_{2\pi}, a_{3\pi}) \in \{a_1, a_3\}$ for every permutation $\pi$ of $\{1, 2, 3\}$, and there exists a permutation $\pi'$ of $\{1, 2, 3\}$ for which $f(a_{1\pi'}, a_{2\pi'}, a_{3\pi'}) \neq f(a_1, a_2, a_3)$.*

*Proof.* We will use Theorem 2. We may suppose, without loss of generality, that $A = \underline{n}$. First, we prove that if one of the conditions $(a)$ or $(b)$ hold for the algebra $(\underline{n}; f)$ then $f$ preserves neither the bounded partial orders nor the binary central relations on $\underline{n}$. We need the following claims.

**Claim.** *Let $\trianglelefteq$ be an arbitrary bounded partial order on $\underline{n}$ with the least element $m$ and the greatest element $M$, then $f$ does not preserve $\trianglelefteq$.*

*Proof of Claim.* If $a \in \underline{n}$, $a \neq m$, $M$, then $f(m, a, M) = m$ or $f(m, a, M) = M$ or $f(m, a, M) = a$. Consider the following $f$-matrices

$$\begin{array}{cc} m & m \\ a & a \\ a & M \\ \hline f(m,a,a) & f(m,a,M) \end{array} \qquad\qquad \begin{array}{cc} m & a \\ a & a \\ M & M \\ \hline f(m,a,M) & f(a,a,M) \end{array}$$

where $f(m, a, a) = f(a, a, M) = a$. If $f(m, a, M) = m$, then the first $f$-matrix shows that $f$ does not preserve $\trianglelefteq$. If $f(m, a, M) = M$, then by the second $f$-matrix $f$ does not preserve $\trianglelefteq$. If $f(m, a, M) = a$, then by $(a)$ or $(b)$ we get that at least one of the elements $f(m,M,a)$, $f(M,m,a)$, $f(M,a,m)$, $f(a,m,M)$, $f(a,M,m)$ is equal to $m$ or $M$. In this case we can get the suitable $f$-matrix by permuting the first three rows of one of the two $f$-matrices above. Now from this $f$-matrix we get that $f$ does not preserve $\trianglelefteq$. The proof of the claim is complete.

**Claim.** *If $\tau$ is an arbitrary binary central relation on $\underline{n}$, then $f$ does not preserve $\tau$.*

*Proof of Claim.* If $c \in \underline{n}$ is a central element of $\tau$ and $a,b \in \underline{n}$ so that $(a,b) \notin \tau$, then consider the following matrices

$$\begin{array}{cc} a & a \\ b & b \\ c & b \\ \hline f(a,b,c) & f(a,b,b) \end{array} \qquad \begin{array}{cc} a & a \\ b & b \\ c & a \\ \hline f(a,b,c) & f(a,b,a) \end{array}$$

where $f(a,b,b) = b$ and $f(a,b,a) = a$. If $f(a,b,c) = a$, then the first $f$-matrix shows that $f$ does not preserve $\tau$. If $f(a,b,c) = b$, then the second $f$-matrix will be used. If $f(a,b,c) = c$, then by $(a)$ or $(b)$ we see that $f(a,c,b)$, $f(b,a,c)$, $f(b,c,a)$, $f(c,a,b)$ or $f(c,b,a)$ is equal to $a$ or $b$. Now we can also get the suitable $f$-matrix by permuting the first three rows of one of the two $f$-matrices above. In this case from this $f$-matrix we get that $f$ does not preserve $\tau$. The proof of the claim is complete.

Now we will prove that if one of the conditions $(a)$ or $(b)$ holds for the algebra $(\underline{n}; f)$, then $f$ does not preserve the nontrivial equivalences on $\underline{n}$.

**Claim.** *If $\rho$ is an arbitrary nontrivial equivalence on $\underline{n}$, then $f$ does not preserve $\rho$.*

*Proof of Claim.* Now there exist elements $a$, $b$, $c \in \underline{n}$ with $a \neq b$, $(a,b) \in \rho$, $(a,c) \notin \rho$.
First, suppose that $(a)$ holds. If $f(a,b,c) = c$, then we can use the following $f$-matrix to show that $f$ does not preserve $\rho$

$$\begin{array}{cc} a & a \\ a & b \\ c & c \\ \hline a & \mathrm{c} \end{array}$$

where $f(a,a,c) = a$. If $f(a,b,c) = a$ or $f(a,b,c) = b$, then by $(a)$ $f(a,c,b)$, $f(b,a,c)$, $f(b,c,a)$, $f(c,a,b)$ or $f(c,a,b)$ equals $c$. In this case we get the suitable $f$-matrix by permuting the first three rows of the $f$-matrix above. From this $f$-matrix we get that $f$ does not preserve $\rho$.

Now we suppose that $(b)$ is true.

(i) First, we suppose that $a \prec b \prec c$. If $f(a,b,c) = c$, then the $f$-matrix above does the job. If $f(a,b,c) = a$, then by $(b)$ $f(a,c,b)$, $f(b,a,c)$, $f(b,c,a)$, $f(c,a,b)$ or $f(c,b,a)$ equals $c$. We get the suitable $f$-matrix by permuting the first three rows of the $f$-matrix above.

(ii) Secondly, we suppose that $c \prec a \prec b$. If $f(c,a,b) = c$ then we get the suitable $f$-matrix by permuting the first three rows of the $f$-matrix above. If $f(c,a,b) = b$, then by $(b)$ $f(c,b,a)$, $f(a,b,c)$, $f(a,c,b)$, $f(b,a,c)$, $f(b,c,a)$ equals $c$. For example, if $f(c,b,a) = c$, then the following $f$-matrix shows that $f$ does not preserve $\rho$

$$\begin{array}{cc} c & c \\ b & a \\ a & a \\ \hline c & a \end{array} .$$

In the remaining cases we get the suitable $f$-matrices by permuting the first three rows of the $f$-matrix above.

(iii) If there do not exist elements $a, b, c \in \underline{n}$ with $a \neq b$, $(a,b) \in \rho$, $(a,c) \notin \rho$ for which $a \prec b \prec c$ or $c \prec a \prec b$ hold, then it is easy to see that $\rho$ has a unique nonsingleton block, namely $\{0, n-1\}$. Now $|A| \geq 4$ and we can suppose that $a = 0$, $b = n-1$ and $\{c_1, \ldots, c_{n-2}\} = \underline{n} \setminus \{a,b\}$.

First, assume $f(a,c_1,c_2) = a$. If $f(b,c_1,c_2) = c_1$, then the following $f$-matrix

$$\begin{array}{cc} a & b \\ c_1 & c_1 \\ c_2 & c_2 \\ \hline a & c_1 \end{array}$$

will be used. If $f(b,c_1,c_2) = b$, then $f(c_2,a,c_1) = c_2$ since the patterns $(b,c_1,c_2)$ and $(c_2,a,c_1)$ are the same with respect to $\preceq$. We need the following $f$-matrices

$$\begin{array}{cc} c_2 & c_2 \\ a & b \\ c_1 & c_1 \\ \hline c_2 & c_1 \end{array} \qquad \begin{array}{cc} c_2 & c_2 \\ a & b \\ c_1 & c_1 \\ \hline c_2 & b \end{array} .$$

If $f(c_2,b,c_1) = c_1$, then the first $f$-matrix shows that $f$ does not preserve $\rho$. If $f(c_2,b,c_1) = b$, then the second $f$-matrix does the job.

Secondly, assume $f(a, c_1, c_2) = c_2$. Now we will use the following $f$-matrices

$$\begin{array}{cc} a & b \\ c_1 & c_1 \\ c_2 & c_2 \\ \hline c_2 & c_1 \end{array} \qquad \begin{array}{cc} a & b \\ c_1 & c_1 \\ c_2 & c_2 \\ \hline c_2 & b \end{array}\ .$$

If $f(b, c_1, c_2) = c_1$, then the first $f$-matrix shows that $f$ does not preserve $\rho$.
If $f(b, c_1, c_2) = b$, then the second $f$-matrix will be used.

The proof of the claim is complete.

From now we show that the algebra $(\underline{n}; f)$ is not functionally complete if $(a)$ and $(b)$ are not satisfied. Further also suppose that $a_1, a_2, a_3 \in \underline{n}$ and $a_1 \prec a_2 \prec a_3$. We have the following three cases:

If $a_i = f(a_1, a_2, a_3) = f(a_{1\pi}, a_{2\pi}, a_{3\pi})$ equalities hold for every permutation $\pi$ of $\{1, 2, 3\}$, then $f$ preserves one of the three binary central relations $\tau_1$, $\tau_2$, $\tau_3$ on $A$ defined below:

For $i = 1$, let the center of $\tau_1$ be $C = \{0, 1, \ldots, n-3\}$ and $(n-2, n-1) \notin \tau_1$,
for $i = 2$, let the center of $\tau_2$ be $C = \{1, 2, \ldots, n-2\}$ and $(0, n-1) \notin \tau_2$,
for $i = 3$, let the center of $\tau_3$ be $C = \{2, 3, \ldots, n-1\}$ and $(0, 1) \notin \tau_3$.

Now let $f(a_{1\pi}, a_{2\pi}, a_{3\pi}) \in \{a_1, a_2\}$ be for every permutation $\pi$ of $\{1, 2, 3\}$ (or let $f(a_{1\pi}, a_{2\pi}, a_{3\pi}) \in \{a_2, a_3\}$ be for every permutation $\pi$ of $\{1, 2, 3\}$), and suppose that there exists a permutation $\pi'$ of $\{1, 2, 3\}$ for which $f(a_{1\pi'}, a_{2\pi'}, a_{3\pi'}) \neq f(a_1, a_2, a_3)$. Then it is easy to show that $f$ preserves the nontrivial equivalence with a unique nonsingleton block, namely $\{0, 1, \ldots, n-2\}$ (or $\{1, 2, \ldots, n-1\}$).

□

**Proposition 5.** *Let $A = \{0, 1, 2\}$ be a linearly ordered set with $0 \prec 1 \prec 2$, and let $f$ be a $\preceq$-pattern function, which is a majority function on $A$. Then the algebra $(A; f)$ is functionally complete iff there exist permutations $\pi, \sigma$ of $A$ for which the values $f(0, 1, 2)$, $f(0\pi, 1\pi, 2\pi)$, $f(0\sigma, 1\sigma, 2\sigma)$ are pairwise distinct.*

*Proof.* Suppose that there exist permutations $\pi, \sigma$ of $A$ for which the values $f(0, 1, 2)$, $f(0\pi, 1\pi, 2\pi)$, $f(0\sigma, 1\sigma, 2\sigma)$ are pairwise distinct. Then the algebra $(A; f)$ is functionally complete. (Let us observe that the proof of this statement is included in the proof of Theorem 4, since in the case $(a)$ of Theorem 4 every $f$-matrix has exactly three elements.)

If $f(0, 1, 2) = f(0\pi, 1\pi, 2\pi)$ for every permutation $\pi$ of $A$, then we obtain that $f$ preserves one of the three binary central relations $\tau_1, \tau_2, \tau_3$ on $A$ defined below:

For $f(0, 1, 2) = 0$ let the center of $\tau_1$ be $\{0\}$, and $(1, 2) \notin \tau_1$,
for $f(0, 1, 2) = 1$ let the center of $\tau_2$ be $\{1\}$, and $(0, 2) \notin \tau_2$,
for $f(0, 1, 2) = 2$ let the center of $\tau_3$ be $\{2\}$, and $(0, 1) \notin \tau_3$.

Now let assume that at least one of the inclusions: $f(0\pi, 1\pi, 2\pi) \in \{0, 1\}$, $f(0\pi, 1\pi, 2\pi) \in \{1, 2\}$, $f(0\pi, 1\pi, 2\pi) \in \{0, 2\}$ holds for every permutation $\pi$ of $A$, and suppose that there exists a permutation $\pi'$ of $A$ for which $f(0\pi', 1\pi', 2\pi') \neq f(0, 1, 2)$. Then it is also easy to observe that $f$ preserves the nontrivial equivalence with unique nonsingleton block, namely $\{0, 1\}$, $\{1, 2\}$ or $\{0, 2\}$. Using Theorem 2, the proof is complete. □

**Theorem 6.** *Let $(A, \preceq)$ be an arbitrary finite poset with $3 \leq |A|$. Let $f$ be a $\preceq$-pattern function, which is a majority function on $A$, and for which there exist permutations $\pi$, $\sigma$ of $\{1, 2, 3\}$ such that the values $f(a_1, a_2, a_3)$, $f(a_{1\pi}, a_{2\pi}, a_{3\pi})$, $f(a_{1\sigma}, a_{2\sigma}, a_{3\sigma})$ are pairwise distinct, then the algebra $(A; f)$ is functionally complete.*

*Proof.* Such an operation $f$ always exists. (For example: $f(x, x, y) = f(x, y, x) = f(y, x, x) = x$, and $f(x, y, z) = x$ if $x, y, z$ are pairwise different). Now it is easy to prove that such operations do not preserve the bounded partial orders, the binary central relations and the nontrivial equivalences on $A$. Applying Theorem 2, the proof is complete. □

**Theorem 7.** *Let $(A; \preceq)$ be an arbitrary finite poset with $3 \leq |A|$. Then for every $k$ with $3 \leq k \leq |A|$ there exists a $k$-ary $\preceq$-pattern function $f$, which is a semiprojection and the algebra $(A; f)$ is functionally complete.*

*Proof.* If $3 \leq k \leq |A|$, then the $k$-ary $\preceq$-pattern function

$$f_k(x_1, x_2, \ldots, x_k) = \begin{cases} x_1 & \textit{if} \text{ the elements } x_1, x_2, \ldots, x_k \text{ are pairwise distinct and} \\ & x_{k-1} \not\prec x_k, \\ x_k & \text{otherwise} \end{cases}$$

is a semiprojection on $A$. By Lemma 7 of [3] $f_k$ has no compatible bounded partial order on $A$.

Let $\tau$ be an arbitrary binary central relation on $A$, let $c \in A$ be a central element of $\tau$, and let $a, b \in A$ be so that $(a, b) \notin \tau$. We will need the following matrices

$$\begin{array}{cc} a & a \\ d & d \\ \vdots & \vdots \\ e & e \\ c & b \\ b & b \\ \hline a & b \end{array} \qquad\qquad \begin{array}{cc} a & a \\ d & d \\ \vdots & \vdots \\ e & e \\ b & b \\ c & b \\ \hline a & b \end{array}$$

where the entries above the line in the first column are pairwise distinct in both $f_k$-matrices.

If $c \not\prec b$, then we will use the first $f_k$-matrix. If $c \prec b$, then the second $f_k$-matrix will work. In both cases we get that $f_k$ does not preserve the relation $\tau$.

Let $\rho$ be an arbitrary nontrivial equivalence, and let $a, b, c \in A$ with $a \not\prec b$, $(a, b) \in \rho$ and $(a, c) \notin \rho$. Now we will use the following $f_k$-matrix to show that $f_k$ does not preserve $\rho$

$$\begin{array}{cc} c & c \\ d & d \\ \vdots & \vdots \\ e & e \\ a & a \\ b & a \\ \hline c & a \end{array}$$

where the entries above the line in the first column of the $f_k$-matrix are pairwise distinct. Using Theorem 2 we get that the algebra $(A; f_k)$ is functionally complete.

□

**Remark.** Let $(A; \preceq)$ be a finite linearly ordered set with $3 \leq |A|$, and let $f$ be a nontrivial $k$-ary $\preceq$-pattern function, which is a semiprojection on $A$. If for any elements $a_1, \ldots, a_k \in A$ with $a_1 \prec \ldots \prec a_k$, and for any permutations $\pi$ of $\{1, \ldots, k\}$ one of the following conditions is satisfied:

(a) $a_i = f(a_{1\pi}, \ldots, a_{k\pi})$, $3 \leq k \leq |A|$, or

(b) $f(a_{1\pi}, \ldots, a_{k\pi}) \in \{a_1, a_2, \ldots, a_{k-2}\}$, $4 \leq k \leq |A|$, or

(c) $f(a_{1\pi}, \ldots, a_{k\pi}) \in \{a_2, a_3, \ldots, a_{k-1}\}$, $4 \leq k \leq |A|$, or

(d) $f(a_{1\pi}, \ldots, a_{k\pi}) \in \{a_3, a_4, \ldots, a_k\}$, $4 \leq k \leq |A|$

then the algebra $(A; f)$ is not functionally complete.

*Proof of Remark.* We may suppose, without loss of generality, that $A = \underline{n}$. If condition (a) holds, then $f$ preserves one of the binary central relation $\tau_1$, $\tau_2$, $\tau_3$ on $A$ defined below:

(1) for $i = 1$, let the center of $\tau_1$ be $C = \{0, 1, \ldots, n-3\}$ and $(n-2, n-1) \notin \tau_1$,

(2) for $1 < i < k$, let the center of $\tau_2$ be $C = \{1, 2, \ldots, n-2\}$ and $(0, n-1) \notin \tau_2$,

(3) for $i = k$, let the center of $\tau_3$ be $C = \{2, 3, \ldots, n-1\}$ and $(0, 1) \notin \tau_3$.

It is also easy to see that if (b) holds, then $f$ preserves the central relation $\tau_1$. If (c) (or (d)) holds, then $f$ preserves the central relation $\tau_2$ (or $\tau_3$). Using Theorem 2, the proof of the remark is complete.

□

Let $(A; \preceq)$ be an arbitrary finite bounded poset with at least three elements. Define the following two operations on $A$:

$$t(x,y,z) = \begin{cases} z & if \quad x \preceq y, \\ x & \text{otherwise}, \end{cases}$$

$$d(x,y,z) = \begin{cases} x & if \quad x \preceq y, \\ z & \text{otherwise}. \end{cases}$$

The operation $t$ is the *ternary order-discriminator*, and $d$ is the *dual order-discriminator*. The algebras $(A; t)$, $(A; d)$ are called *order-discriminator algebras*. In [12] we proved that the order-discriminator algebras $(A; t)$ and $(A; d)$ are functionally complete. The following theorem is a generalization of this result.

**Theorem 8.** *If $(A; \preceq)$ is an arbitrary finite poset with at least three elements, then the order-discriminator algebras $(A; t)$ and $(A; d)$ are functionally complete.*

*Proof.* It is sufficient to prove that $t$ and $d$ do not preserve the relations $(a)$, $(b)$, and $(c)$ in Theorem 2.

(a) Let $\trianglelefteq$ be an arbitrary bounded partial order on $A$ with the least element $m$ and the greatest element $M$. Now we show that the operations $t, d$ do not preserve the bounded partial order $\trianglelefteq$ on $A$. Let $a \in A$ be an arbitrary element different from $m$ and $M$. The following two $t$-matrices and two $d$-matrices will be used

$$\begin{array}{cc} m & m \\ m & a \\ M & M \\ \hline M & m \end{array} \qquad \begin{array}{cc} a & M \\ m & M \\ m & m \\ \hline a & m \end{array} \qquad\qquad \begin{array}{cc} a & M \\ a & a \\ m & m \\ \hline a & m \end{array} \qquad \begin{array}{cc} a & a \\ a & M \\ m & m \\ \hline a & m \end{array} .$$

If $a \prec m$ then the first $t$-matrix, if $a \not\prec m$ then the second $t$-matrix shows that $t$ does not preserve $\trianglelefteq$. If $a \prec M$ then the first $d$-matrix, if $a \not\prec M$ then the second $d$-matrix shows that $d$ does not preserve $\trianglelefteq$.

(b) Let $\tau$ be an arbitrary central relation on $A$, and let $a, b, c \in A$ so that $a \neq b$, $(a, b) \notin \tau$ and $c$ is a central element of $\tau$. We may suppose that $a \not\prec b$. Consider the following $t$-matrix and $d$-matrix

$$\begin{array}{cc} a & c \\ b & c \\ c & b \\ \hline a & b \end{array} \qquad\qquad \begin{array}{cc} a & a \\ a & c \\ c & b \\ \hline a & b \end{array} .$$

The first $t$-matrix shows that the operation $t$ does not preserve $\tau$. If $a \not\preceq c$ then by the $d$-matrix we see that the operation $d$ does not preserve $\tau$. If $a \preceq c$, then by permuting the first two rows of the $d$-matrix we get again that $d$ does not preserve $\tau$.

(c) Let $\varepsilon$ be an arbitrary nontrivial equivalence on $A$, and let $a, b, c \in A$ so that $(a, b) \in \varepsilon$ and $(a, c) \notin \varepsilon$. We will need the following two $t$-matrices and two $d$-matrices:

$$\begin{array}{cc} a & b \\ a & a \\ c & c \\ \hline c & b \end{array} \quad \begin{array}{cc} a & a \\ a & b \\ c & c \\ \hline c & a \end{array} \qquad\qquad \begin{array}{cc} a & b \\ a & a \\ c & c \\ \hline a & c \end{array} \quad \begin{array}{cc} a & a \\ a & b \\ c & c \\ \hline a & c \end{array} .$$

If $a \prec b$, then by the first $t$-matrix, if $a \not\prec b$, then by the second $t$-matrix we get that the operation $t$ does not preserve the relation $\varepsilon$. If $a \prec b$, then the first $d$-matrix, if $a \not\prec b$, then the second $d$-matrix does the job. In all cases we see that the operations $t$ and $d$ do not preserve $\varepsilon$. □

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