On a maximum-minimum problem and its connexion with the roots of equations.

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1. In the present communication I intend to give a short account on my investigations concerning a maximum-minimum problem and its applications.

The origin of these investigations was the following theorem, stated and proved by P. J. *Heawood*[1]):

(*A*) *If* $f(z)$ *is a polynomial of the degree* n, *and if* $f(+1)=o$ *and* $f(-1)=o$, *then* $f'(z)=o$ *has a root in the circle, whose centre is the origin and whose radius is* $\varrho = ctg\frac{\pi}{n}$.

Later on my attention was called to a paper of J. H. *Grace*[2]), where a general theorem concerning the roots of „apolar" equations is proved, which contains the before-mentioned theorem, as a special case.

Recently, M. G. *Szegő*[3]) published various interesting applications of this theorem.

2. Before I proceed to the discussion of my own results obtained in this subject, I wish to point out, how we are led, in attempting to treat the question in theorem (A), quite naturally to the investigation of a maximum-minimum problem.

I state the question in the following form: Let $f(z)$ be a polynomial of the degree n and such that $f(z)$ assumes the same values for $z=1$ and $z=-1$. It is to be determined the smallest circle, whose centre is the origin and which contains at least one root of the derived equation $f'(z)=0$.

Let us denote the roots of $f'(z)=0$ by $z_1, z_2, \dots z_{n-1}$ (that is $f'(z)=C(z-z_1)(z-z_2)\dots(z-z_{n-1})$), then the assump-

[1]) P. J. *Heawood*, Geometrical relations between the roots of $f(x)=0$, $f'(x)=0$, Quarterly Journal of Mathematics *38* (1907), pp. 84—107.

[2]) J. H. *Grace*, The zeros of a polynomial, Proceedings of the Cambridge Philosophical Society *11* (1900—1902), pp. 352—357.

[3]) G. *Szegő*, Bemerkungen zu einem Satz von J. H. Grace über die Wurzeln algebraischer Gleichungen, Mathematische Zeitschrift *13* (1922), pp. 28—55.

tion, that $f(z)$ assumes the same values for $z=+1$ and $z=-1$, is evidently equivalent to the following equation:

$$(1)\quad (f)+1)-f(-1)=\int_{-1}^{+1} f(z)dz=C\int_{-1}^{+1}(z-z_1)(z-z_2)\ldots(z-z_{n-1})=0$$

or

$$(1')\quad z_1 z_2 \ldots z_{n-1}+\frac{1}{3}\sum z_1 z_2 \ldots z_{n-3}+\frac{1}{5}\sum z_1 z_2 \ldots z_{n-5}+\ldots=0.$$

In this way our problem is reduced to the following one: Consider all the systems of (complex) values $(z_1, z_2, \ldots z_n)$, wich satisfy the symmetric, multilinear relation

$$(2)\qquad c_0 z_1 z_2 \ldots z_n + c_1 \Sigma z_1 z_2 \ldots z_{n-1} + \ldots + c_{n-1} \Sigma z_1 + c_n = o$$

Denote by $m(|z_1|, \ldots |z_n|)$ the smallest amongst the moduli $|z_1|, |z_2|, \ldots |z_n|$. Now we have

I. to show, that there is an upper limit M, depending on the coefficients $c_o, c_1, \ldots c_n$ and such that for all the systems $(z_1, \ldots z_n)$, which satisfy (2),

$$(3)\qquad m(|z_1|, \ldots |z_n|) \leqq M$$

and that there is at least one system, for which $m(|z_1|, \ldots |z_n|)$ reaches this upper limit (viz. M is the maximum of $m(|z_1|, \ldots |z_n|)$);

II. to determine

$$(4)\qquad \text{Max. } m(|z_1|, \ldots |z_n|);$$

III. to determine all the systems of values $(z_1, \ldots z_n)$, for which this maximum is reached.

3. The first part of the question may be treated by most elementary considerations.

The second part, viz. the determination of Max. $m(|z_1|, \ldots |z_n|)$ was solved in the case of the special multilinear relation (1′) by *Heawood*. As to the general symmetric, multilinear relation (2), the solution is implicitely contained in the theorem (on apolar equations) of *Grace*.

The third part of the problem, the determination of all the systems $(z_1, \ldots z_n)$, for which the maximum is reached, is, so far as I am aware, neither by the mentioned authors, nor by others attempted.

My investigations enabled me to treat completely the question and the following theorems contains the full discussion of the general problem.

As to the proofs, further details and applications see my communication (in hungarian language), which appears next in the Mathematikai és fizikai lapok, Budapest, 1922.

The general solution of the maximum-minimum problem.

4. Before entering upon the details of the problem, it is necessary to mention some properties of a symmetric, multilinear function.

A symmetric, multilinear function is an expression of the form

$$(5)\quad S(z_1, z_2, \dots z_n) = c_0 z_1 z_2 \dots z_n + c_1 \Sigma z_1 z_2 \dots z_{n-1} + \dots + c_{n-1} \Sigma z_1 + c_n$$

where $c_0, c_1, \dots c_n$ are constants.

If the function S is ordered, corresponding to certain groups of the variables $z_1, \dots z_n$, it will assume the following forms:

$$S = z_1 S_{1,0}(z_2, \dots) + S_{1,1}(z_2, \dots z_n)$$

$$(6)\quad = z_1 z_2 S_{2,0}(z_3, \dots z_n) + (z_1 + z_2) S_{2,1}(z_3, \dots z_n) + S_{2,2}(z_3, \dots z_n)$$

$$\dots\dots\dots$$

$$= \sum_{l=0}^{k} S_{kl}(z_{k+1}, \dots z_n) . s_{kl}(z_1, \dots z_k)$$

where

$$(6')\quad S_{ki}, (z_{k+1} \dots z_n) = \sum_{j=0}^{n-k} c_{i+j} s_{n-k\,j}(z_{k+1} \dots z_n) ; (i=0, 1, \dots k)$$

and $s_j(z_1, \dots z_h)$ denotes the elementary symmetric function of the variables $z_1, \dots z_h$.

If the values of the variables $z_1, \dots z_h$ are all equal, and their common value is denoted by ζ, then the function S becomes

$$(7)\quad G(\zeta) = S(\zeta, \zeta, \dots \zeta) = \zeta^n + \binom{n}{1} c_1 \zeta^{n-1} + \dots + c_n$$

The polynomial $G(\zeta)$ of the degree n shall be denominated the *adjungated* polynomial of the symetric, multilinear function S.

If amongst the variables $z_1, \dots z_n$ there are $n-k$, i. e. $z_{k+1}, \dots z_n$ equal to ζ, then the function S will be transformed in the following symmetric, multilinear function of the k variables $z_1, \dots z_k$:

$$(8)\quad S(z_1, \dots z_k, \zeta, \dots \zeta) = \sum_{l=0}^{k} H_l^{(k)}(\zeta) . s_{k-l}(z_1 \dots z_k)$$

where

$$H_1^{(k)}(\zeta) = \frac{1}{n(n-1)\dots(n-k+1)} G^{(k)}(\zeta),$$

$$H_2^{(k)}(\zeta) = \frac{1}{n(n-1)\dots(n-k+2)} \left[G^{(k-1)}(\zeta) - \frac{\zeta}{n-k+1} G^{(k)}(\zeta)\right],$$

(8′) ....

$$H_k^{(k)}(\zeta) = G(\zeta) - \binom{k}{1}\frac{\zeta}{n} G'(\zeta) + \binom{k}{2}\frac{\zeta^2}{n(n-1)} G''(\zeta) - \dots +$$

$$+ (-1)^k \frac{\zeta}{n(n-1)\dots(n-k+1)} G^{(k)}(\zeta)$$

$$= \left(1 - \frac{\zeta}{n-k+1}\frac{d}{d\zeta}\right)\left(1 - \frac{\zeta}{n-k+2}\frac{d}{d\zeta}\right)\dots\left(1 - \frac{\zeta}{n}\frac{d}{d\zeta}\right) G(\zeta)$$

The adjungated polynomial of $S_{k,k}(z_{k+1}, \dots z_n)$ is $H_k^{(k)}(\zeta)$.

In the following investigation I consider only such systems of values $(z_1, \dots z_n)$, which satisfy the relation

(9) $$S(z_1, \dots z_n) = c_0 z_1 z_2 \dots z_n + c_1 \Sigma z_1 z_2 \dots z_{n-1} + \dots + c_n = 0.$$

An element of such a system $(z_1 \dots z_n)$ shall be called for shortness the absolutely greatest (least) element, if its modulus is not less (greater) than the modulus of any other element of the system, and its modulus shall be denoted by $M(|z_1|, \dots |z_n|)$ $[m(|z_1|, \dots |z_n|)]$.

A system $(z_1, \dots z_n)$, whose elements are all equal, shall be called a *coincident* system.

Finally, a system, whose elements are all different, shall be called a *dispersed* system.

5. We are now able to state with clearness the problem and its general solution:

It is to be determined $P =$ Max. $m(|z_1|, \dots |z_n|)$ for all the systems $(z_1, \dots z_n)$ subject to the relation $S = 0$.

There are to be determined all the systems $(z_1^*, \dots z_n^*)$, for which the maximum is reached. So these latter systems $(z_1^*, \dots z_n^*)$, which we shall call *the maximal systems corresponding in the relation* $S = 0$, have the two characteristic properties:

(10) $$S(z_1^*, \dots z_n^*) = 0,$$
$$m(|z_1^*|, \dots |z_n^*|) = P.$$

The determination of Max. $m(|z_1|, \dots z_n|)$ will be furnished by the theorem

(I.) *Amongst the maximal systems corresponding to the relation* $S=0$ *there is always one coincident at least*, the immediate consequence of which is theorem

(II.) *If* $\zeta_1, \zeta_2, \ldots \zeta_n$ *denote the roots of the adjungated equation* $G(z)=0$ (7.), then

$$(11) \qquad P = \text{Max.}\, m(|z_1|, \ldots |z_n|) \leqq M(|\zeta_1|, \ldots |\zeta_n|)$$

that is to say, *the abs. least element of a system* $(z_1, \ldots z_n)$, *subject to the relation* $S=0$ *is not greater than the abs. largest root of the adjungated equation* $G(z)=0$.

In other words, the groups of points $(z_1, \ldots z_n), (\zeta_1, \ldots \zeta_n)$ in the Argand diagramm cannot be separated by any circle, whose centre is the origin.

6. The relation $S=0$ (9.) may be obviously written in the following forms

$$(12) \qquad s_n - \frac{\sigma_1 s_{n-1}}{\binom{n}{1}} + \frac{\sigma_2 s_{n-2}}{\binom{n}{2}} - \ldots + (-)^n \sigma_n = 0$$

(where s_k, resp. σ_k denote the elementary symmetric functions of the quantities $z_1, \ldots z_n$, resp. $\zeta_1, \ldots \zeta_n$), or

$$(13) \qquad \sum_i (z_1 - \zeta_{i_1})(z_2 - \zeta_{i_2}) \ldots (z_n - \zeta_{i_n}) = 0$$

In consequence of (13.) it is evident, that the relation $S=0$ (9.) is invariant with respect to the translation of the origin, so that the last form of theorem (II.) may be immediately generalised, as follows.

(III.) *If the two systems of values* $(z_1, \ldots z_n)$, $(\zeta_1, \ldots \zeta_n)$ *are connected by the relation* $S=0$ (9.), *or by* (13.), *the two corresponding groups of points in the Argand diagramm cannot be separated by any circle (or straight line).*

This is the form, in which the solution of the problem is stated by G. H. *Grace* (l. c.).

The discussion of the extremal systems.

7. In order to discuss the configuration of the extremal systems, we have to investigate the series of functions (defined by (6), (6'))

$$S(z_1, \ldots z_n), \quad S_{1,1}(z_2 \ldots z_n) \quad S_{2,2}(z_3, \ldots z_n), \ldots$$

Suppose, that in the series (11,) $S_{k,k}$ is the first term, which does not vanish for any of the maximal systems corresponding to the relation $S=0$, so that there is a maximal system $(z_1^*, \dots z_n^*)$, for which

$$S=0, \; S_{1,1}=0, \dots \; S_{k-1,k-1}=0$$

Then it may be proved (see my Hungarian communication, l. c.), that amongst the absolutely largest roots of the adjungated equation $G(z)=0$ there is at least one, for which

$$(15) \qquad G(\zeta)=0, \; H_1^{(1)}(\zeta)=0, \dots \; H_{k-1}^{(k-1)}(\zeta)=0$$

and from the definition of $H_1^{(1)}(\zeta)$ given by (8), (8′) immediately follows, that

$$(16) \qquad G(\zeta)=0, \; G'(\zeta)=0, \dots \; G^{(k-1)}(\zeta)=0,$$

that is, ζ is a root with the multiplicity k for the equation $G(\zeta)=0$. So we get the theorem

(IV.) *If in the series*

$$(17) \qquad S, \; S_{1,1}, \; S_{2,2}, \dots$$

$S_{k,k}$ *is the first term, which does not vanish for any of the maximal systems corresponding to the relation* $S=0$, *then amongst the abs. largest roots of the adjungated equation* $G(\zeta)=0$ *there is at least one, whose multiplicity is k, and there is no root with greater multiplicity. Amongst the elements of the maximal systems the value of* $k-1$ *elements may be taken arbitrarily* (subject only to the inequality $|z_k| \leq P$) *and the other* $n-k+1$ *elements must coincide with such an abs. largest root of* $G(z)=0$, *whose multiplicity is k.*

This theorem leads to the following classification of the maximal systems:

(*A*) *Suppose, that amongst the abs largest roots of the adjungated equation* $G(z)=0$ *there are multiple roots and that the highest multiplicity is k. Denote by P the modulus of the abs. largest roots of* $G(z)=0$. *Then*

(A_1) *If the roots of* $G(z)$ *are not all bying on the circle* $|z|=P$ *then amongst the elements of a maximal system* $\varkappa \leq k-1$ *elements may be chosen arbitrarily and the other* $n-\varkappa \geq n-k+1$ *elements must coincide with such an abs. largest root of* $G(z)=0$, *whose* *multiplicity is at least* $\varkappa+1$. *In this case there are no dispersed maximal systems.*

(A_2) *If all the roots of* $G(z)=0$ *are on the circle* $|z|=\varrho$, *then besides of the maximal systems defined in* (A_1), *there are dispersed maximal systems such that* $n-1$ *of their elements may be chosen on the circle* $|z|=P$ *arbitrarily.*

(B) *Suppose, that all the abs. largest roots of* $G(z)=0$ *are simple roots. Then*

(B_1) *If the equation* $G(z)=0$ *has all its roots on the circle* $|z|=P$, *then the maximal systems are such, that* $n-1$ *of its elements may be taken on the circle* $|z|=P$ *arbitrarily.*

(B_2) *If the roots of* $G(z)=0$ *are not all lying on the circle* $|z|=P$, *then the maximal systems are only in finite number. In this case every maximal system is coinciding with an abs. largest root of* $G(z)=0$.

In the special case, that $G(z)=0$ has only one abs. largest root, the maximum problem has a *unique* solution, and the only maximal system coincides with the abs. largest root of $G'(z)=0$.

Corresponding theorems may be stated in the case of the minimum problem.