On the Montmort-Moivre Problem.

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A few years ago I considered with *R. Fiedler* the same generalisation of the *Montmort—Moivre* Problem as is presented in Professor *Kürschák*'s preceding paper[1]; but as we used other methods, our results were expressed differently, namely, by definite integrals and sums. In putting them equal to the results of the foregoing paper, the obtained formulae might present some interest. To make the comparison easier, let us adopt the same designations as are used therein.

Being given n urns, the first of which contains the numbers $0, 1, 2, \ldots s_1-1$, the second $0, 1, 2, \ldots s_2-1$, and so on, the n-th containing $0, 1, 2, \ldots s_n-1$. A number x is drawn from each urn; required the probability that the drawn numbers $x_1, x_2, \ldots x_n$ satisfy the equation:

$$a_1 x_1 + a_2 x_2 + \ldots a_n x_n = k,$$

where $a_1, a_2, \ldots a_n$ are given integers.

The total number of the possible cases is obviously equal to $s_1 s_2 \ldots s_n$. To determine the number F of the favourable cases let us start first from the discontinuous factor

$$\psi_1 = \frac{2}{\pi} \int_0^{1/2\pi} \cos (X-k) 2\varphi \, . \, d\varphi$$

If X is an integer different from k, the factor ψ_1 is equal to zero; if $X=k$, then we have $\psi_1 = 1$. Accordingly F is obtained by putting $X = a_1 x_1 + a_2 x_2 + \ldots a_n x_n$ and by taking the sum for every positive value of the variables x_i. Hence,

[1]) Acta Litterarum ac Scientiarum, Tom. I. p. 139—143.

$$F = \sum_{x_1=0}^{s_1} \sum_{x_2=0}^{s_2} \dots \sum_{x_n=0}^{s_n} \psi_1 \qquad ^{2})$$

Let us remark that $\cos (a_1 x_1 + \dots + a_n x_n - k) 2\varphi$ is the real part of

$$e^{-2 i \varphi k} \prod_{\nu=1}^{n+1} e^{2 i \varphi a_\nu s_\nu}$$

and that

$$\sum_{x_\nu=0}^{s_\nu} e^{2 i \varphi a_\nu x_\nu} = \frac{e^{2 i \varphi a_\nu s_\nu} - 1}{e^{2 i \varphi a_\nu} - 1} = e^{i \varphi (a_\nu s_\nu - a_\nu)} \frac{\sin a_\nu s_\nu \varphi}{\sin a_\nu \varphi}$$

Therefore F is the real part of

$$\frac{2}{\pi} \int_0^{1/2 \pi} e^{i \varphi (\Sigma a_\nu s_\nu - \Sigma a_\nu - 2k)} \prod_{\nu=1}^{n+1} \frac{\sin a_\nu s_\nu \varphi}{\sin a_\nu \varphi} d\varphi$$

so that F will be equal to

$$(1) \qquad F = \frac{2}{\pi} \int_0^{1/2 \pi} \cos (\Sigma a_\nu s_\nu - \Sigma a_\nu - 2k) \varphi \prod_{\nu=1}^{n+1} \frac{\sin a_\nu s_\nu \varphi}{\sin a_\nu \varphi} d\varphi$$

Some of the properties of these numbers can be deduced from the preceding formula; e. g., if we consider F as a function of k, it follows that F is a symmetrical function; indeed we have

$$F(k) = F(\Sigma a_\nu s_\nu - \Sigma a_\nu - k) \quad ;$$

This symmetry is relative to $k_0 = {}^1/_2 \Sigma a_\nu s_\nu - {}^1/_2 \Sigma a_\nu$; if k_0 is an integer we have:

$$F({}^1/_2 \Sigma a_\nu s_\nu - {}^1/_2 \Sigma a_\nu) = \frac{2}{\pi} \int_0^{1/2 \pi} \prod_{\nu=1}^{n+1} \frac{\sin \varphi a_\nu s_\nu}{\sin \varphi a_\nu} d\varphi$$

If k_0 is not an integer, we have two consecutive values of F, which are equal:

$$F(k_0 - {}^1/_2) = F(k_0 + {}^1/_2) = \frac{2}{\pi} \int_0^{1/2 \pi} \cos . \varphi \sum_{\nu=1}^{n+1} \frac{\sin \varphi a_\nu s_\nu}{\sin \varphi a_\nu} d\varphi$$

From the circumstances of the problem it results, that if we put $k < 0$ or $k > \Sigma a_\nu s_\nu - \Sigma a_\nu$ then we shall have $F = 0$ More-

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[2]) In this paper, as in the Calculus of Finite Differences, we will understand by $\sum_{x=1}^{n} f(x)$ the sum $f(1) + f(2) + \dots f(n-1)$; i. e., the upper limit is not included. The same remark applies to the product.

over it follows that $F(0) = F(\Sigma a_\nu s_\nu - \Sigma a_\nu) = 1$. And if we have for every value of μ, $k < a_\mu s_\mu$ or $k > \Sigma a_\nu s_\nu - \Sigma a_\nu - a_\mu s_\mu$ then the number F will be independent of the numbers $s_1, s_2, \ldots s_n$; and formula (1) will be the expression of Professor Kürschák's number: [k] This latter is the coefficient of x^k in the expansion of:

$$\prod_{\nu=1}^{n+1} \frac{1}{1-x^{a_\nu}}$$

Therefore, putting the quantity $\omega = c \prod_{\nu=1}^{n+1} a_\nu > k$ into (1) for every $a_\nu s_\nu$, the value of the latter will not change, and it results

$$F = \frac{2}{\pi} \int_0^{1/2\pi} \cos(n\omega - \Sigma a_\nu - 2k)\varphi \sin^n \omega\varphi \prod_{\nu=1}^{n+1} \frac{d\varphi}{\sin a_\nu \varphi}$$

If instead of ψ_1 we start from the discontinuous factor

$$\psi_2 = \frac{2}{\pi} \int_0^{1/2\pi} \cos 2X\varphi \cos 2k\varphi \, d\varphi \qquad \text{or from}$$

$$\psi_3 = \frac{2}{\pi} \int_0^{1/2\pi} \sin 2X\varphi \sin 2k\varphi \, d\varphi$$

we obtain in the same manner:

$$(2)\quad F = \frac{2}{\pi} \int_0^{1/2\pi} \cos 2k\varphi \cos(\Sigma a_\nu s_\nu - \Sigma a_\nu)\varphi \prod_{\nu=1}^{n+1} \frac{\sin a_\nu s_\nu \varphi}{\sin a_\nu \varphi} d\varphi$$

and

$$(3)\quad F = \frac{2}{\pi} \int_0^{1/2\pi} \sin k\varphi \sin(\Sigma a_\nu s_\nu - \Sigma a_\nu)\varphi \prod_{\nu=1}^{n+1} \frac{\sin a_\nu s_\nu \varphi}{\sin a_\nu \varphi} d\varphi$$

Moreover, instead of using discontinuous integrals we may employ the following discontinuous sums

$$\chi_1 = \frac{1}{p} \sum_{\mu=0}^{p} \cos \frac{2\pi\mu(X-k)}{p} \qquad \text{or}$$

$$\chi_2 = \frac{2}{p} \sum_{\mu=0}^{p} \cos \frac{2\pi\mu k}{p} \cos \frac{2\pi\mu X}{p} \qquad \text{or}$$

$$\chi_3 = \frac{2}{p} \sum_{\mu=0}^{p} \sin \frac{2\pi\mu k}{p} \sin \frac{2\pi\mu X}{p}$$

p being an odd integer greater than every value of $X+k$; if X is a positive integer different from k, then $\chi_1=\chi_2=\chi_3=0$; if on the contrary X is equal to k, then $\chi_1=\chi_2=\chi_3=1$.

The same method as before conducts to the following results:

$$\text{(4)}\qquad F=\frac{1}{p}\sum_{\mu=0}^{p}\cos\left(\Sigma a_\nu s_\nu-\Sigma a_\nu-2k\right)\frac{\pi\mu}{p}\prod_{\nu=1}^{n+1}\frac{\sin a_\nu s_\nu\frac{\pi\mu}{p}}{\sin a_\nu\frac{\pi\mu}{p}}$$

$$\text{(5)}\qquad F=\frac{2}{p}\sum_{\mu=0}^{p}\cos 2k\frac{\pi\mu}{p}\cos\left(\Sigma a_\nu s_\nu-\Sigma a_\nu\right)\frac{\pi\mu}{p}\prod_{\nu=1}^{n+1}\frac{\sin a_\nu s_\nu\frac{\pi\mu}{p}}{\sin a_\nu\frac{\pi\mu}{p}}$$

$$\text{(6)}\qquad F=\frac{2}{p}\sum_{\mu=0}^{p}\sin 2k\frac{\pi\mu}{p}\sin\left(\Sigma a_\nu s_\nu-\Sigma a_\nu\right)\frac{\pi\mu}{p}\prod_{\nu=1}^{n+1}\frac{\sin a_\nu s_\nu\frac{\pi\mu}{p}}{\sin a_\nu\frac{\pi\mu}{p}}$$

The quantity F as given by (1), (2), (3), ... (6) is equal to the expression (7) of the preceding paper.

Let us now consider the particular case in which $a_1=a_2=\ldots=a_n=1$. Putting F equal to the result found by *Kürschák* we have according to (1)

$$\text{(7)}\qquad \frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\cos\left(\Sigma s_\nu-n-2k\right)\varphi\prod_{\nu=1}^{n+1}\frac{\sin s_\nu\varphi}{\sin\varphi}\,d\varphi=$$

$$=\sum_{m=0}(-1)^m\binom{n+k-1-s_{\alpha_1}-\ldots s_{\alpha_m}}{n-1}$$

In this formula every combination of the m-th order of the numbers $1, 2, \ldots n$ must be taken for $\alpha_1, \alpha_2, \ldots \alpha_m$; omitting the terms in which $s_{\alpha_1}+s_{\alpha_1}+\ldots+s_{\alpha_m}>k$.

Starting from our formulae (2), ... (6) we get equalities similar to (7); the right side remaining the same. Hence the obtained formulae may be considered as the solutions of the foregoing definite integrals and sums.
