Some notes on the notion of capacity in potential theory.*)

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1. Introduction. The study of the problem of DIRICHLET for general regions and for continuous boundary values has, by recent investigations, been reduced to an examination of the character of the boundary in the neighborhood of each of its points. Let T denote a domain (or open continuum) of space, and let $T_1, T_2, \ldots$ denote an infinite sequence of domains in T for each of which the DIRICHLET problem is possible, each including the preceding, and such that each point in T lies in a domain of the sequence. Let $f(p)$ denote a continuous function of the position of the point p on the boundary t of T, and $F(P)$ a function continuous throughout space, and coinciding with $f(p)$ on t. If u_n is the function harmonic in T_n, assuming the same boundary values as $F(P)$, then the sequence $u_1, u_2, \ldots$ converges to a harmonic limit u, uniformly in any closed region in T.[1]) The function u is independent of the set of regions $T_1, T_2, \ldots$ and of the continuous extension $F(P)$ of the assigned boundary values $f(p)$.[2]) There are points of t at which u approaches $f(p)$, no matter how this continuous function is chosen. These are called *regular* boundary points. For some regions there exist boundary points for which u does not approach the given boundary values for all continuous $f(p)$. Such points are called *exceptional* boundary points. If the

*) Lecture delivered at the meeting on June 8, 1928 of the Mathematical Seminary of the University of Szeged.

[1]) KELLOGG, *Proc. Amer. Acad.*, Vol. 58 (1923) p. 528—29.

[2]) N. WIENER, *Journ. of Math. and Phys. of the Mass. Inst. of Tech.*, vol. 3 (1924) p. 25.

boundary values are given by $1/\overline{pQ}$, where Q is a fixed interior point of T, $G(P, Q) = 1/\overline{PQ} - u$ is a generalized function of GREEN for T, and the regular boundary points are those at which $G(P, Q) \to 0$. All other boundary points are exceptional.[3]) WIENER[4]) has given a criterion for the regular or exceptional character in terms of the notion of *capacity* of a set of points. Let e denote any bounded set of points. The set e, together with its limit points, may have as complement several domains, but the complement will certainly contain an infinite domain T whose whole boundary lies in e. The function u, harmonic in T and vanishing at infinity, corresponding to the boundary values 1, in the manner indicated above, is called the *conductor potential* of the set e. The total charge producing this potential, given by GAUSS' integral, is called the *capacity* of the set e. Obviously it is never negative.

2. The Bounds of Harmonic Functions. Consider first a domain T bounded by a smooth surface t. Let U be harmonic in T and continuous in $T+t$, and let M denote its maximum. Then for any $\alpha > 0$, the set of points e of t at which $U > M - \alpha$ contains all the points of a surface in a neighborhood of one of its points, and it is a simple matter to show that e has positive capacity. We now establish a generalization of this fact:

Theorem I. *Let T be any domain of space, whose boundary is a bounded, non-empty set. Let U be bounded and harmonic in T. If M denotes the least upper bound of U, the set e of boundary points of T at which* $\limsup U > M - \alpha$ *is, for any positive α, a set of positive capacity.*

Suppose the theorem were untrue, and that the capacity of the set e in question were 0. Let τ denote the infinite domain in the complement of e whose whole boundary lies in e, and $\tau_1, \tau_2, \ldots$ a sequence of nested domains whose limit is τ, for each of which the DIRICHLET problem is possible. If u_n is the conductor potential of τ_n, the function

$$M - \alpha + u_n \alpha - U$$

is harmonic in the domain common to τ_n and τ, and has a non-negative lower limit everywhere on the boundary of this domain

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3) KELLOGG, *Proc. Nat. Acad. Washington*, vol. 12 (1926), p. 398.

4) Loc. cit. 2).

Hence it is nowhere negative. This is true in the limit as n becomes infinite. But if the capacity of e were 0, $\lim_{n\to\infty} u_n$ would be 0 everywhere except on e and we should have

$$M-\alpha-U\geqq 0$$

throughout T. Thus $M-\alpha$ would be an upper bound of U, contrary to the hypothesis. It follows that the capacity of e must be positive.

3. Removable Singularities. The notion of capacity makes possible a complete characterization of the sets of points which are the seats of at most removable singularities of harmonic functions. The facts are given in the two following theorems.[5])

Theorem II. *Let T be any domain whose boundary is a bounded set of points, and let B be any portion of the boundary with the properties.*

a) *the set of points $T'=T+B$ is a domain (open continuum), and*

b) *the portion of B in any closed region in T' has the capacity* 0.

Then any function U bounded and harmonic in T, may be so defined at the points of B as to be harmonic in T'.

Let Q be any point of B. By property a), it is the center of a sphere σ, lying in T'. On the surface of σ, U is bounded and continuous, except at the points of B. There is therefor a function V, bounded and harmonic in σ, and approaching on the surface the same values as U at all points not in B.[6]) Then in the portion of T within σ, $V-U$ is bounded, and approaches the boundary values 0, except at the points of B in and on σ, that is, by property b), except at the points of a set of capacity 0. It follows from Theorem I that $V-U=0$ in the portion of T within σ. Hence if we modify U so as to be equal to V within σ, it will be harmonic in a neighborhood of Q. Thus the singularity of U at any point of B is removable.

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[5]) The first of these is a generalization of theorem VII given in my paper in the *Proc. Nat. Acad. Washington*, loc. cit. The second is the correct form of theorem VIII in the same paper, there incorrectly stated. The inaccuracy was kindly pointed out to me by Dr. Vasilesco, and I am grateful for this opportunity to set the matter right.

[6]) See Lemma III, *Proc. Nat. Acad. Washington*, loc. cit.

Theorem III. *Conversely, if B is any set with the property* a), *and if any function, bounded and harmonic in T can have at most removable singularities at the point of B, then B has the property* b).

Let B' denote the portion of B in any closed region in T', and let u denote the conductor potential of B'. This function is bounded and harmonic everywhere except at the points of B' and their limit points. But B' is closed and thus the only singularities of u belong to B, and so, by hypothesis are removable. If u is suitably redefined at the points of B', it becomes harmonic everywhere, and as it vanishes at infinity, it vanishes identically. Hence the capacity of B' is 0, as was to be proved.

4. A Suspected Theorem of Uniqueness. In the introductory section, it was indicated how to given continuous boundary values always corresponds a function, harmonic in T. This function is always bounded, and assumes the given boundary values at every regular boundary point. The method of determining this function cannot lead to a different result. The question suggests itself *could any other method lead to a different function, bounded and harmonic in T, and assuming the same continuous boundary values at every regular point?* Confining ourselves to domains whose boundary set is bounded, this question would be settled if the following statement were established:

A) *There is no function other than* 0 *which is bounded and harmonic in T and which approaches* 0 *at every regular boundary point.*

As a possible contribution to the problem of settling the validity or falsity of this statement, we shall prove that it is equivalent to the following:

B) *Any bounded closed set of points of positive capacity contains at least one regular point,* that is, a regular point of the boundary of the infinite domain whose whole boundary is contained in the set.

Let U denote a bounded function, harmonic in T, approaching 0 at every regular boundary point. If it is not identically 0 either U or $-U$ will have a positive least upper bound M. Consider the first case. The set e at which $\limsup U \geqq M/2$, is bounded, closed, and of positive capacity, by Theorem I. Hence it follows from B) that e contains a regular point, at which $\lim U = 0$, and we have a contradiction. The assumption that either $\limsup U$ or $\limsup (-U)$ is positive is untenable, and thus A) follows from B).

Now let e be a bounded, closed set, of positive capacity, and let T denote the infinite domain of the complement of e, whose boundary is contained in e. If the boundary of T contains no regular points, both 0 and the conductor potential u of e approach 0 at all regular boundary points of T, and it follows from A) that $u = 0$ in T, and hence that the capacity of e is 0. Thus the assumption that e has no regular points is in contradiction with A) and hence B) follows from A).

(Received June 29, 1928)