A remark on the dynamical rôle of POINCARÉ's last geometric theorem.*)

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Consider a dynamical system with one degree of freedom; the equations of motion in the canonical form of HAMILTON are:

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}, \quad \frac{dq}{dt} = \frac{\partial H}{\partial p}$$

where $H = H(p, q)$ is a function of p and q. From the condition that the energy $H(p, q)$ is constant, the solution can be obtained by a quadrature, and this case does not offer any especial interest.

For the case of two degrees of freedom assume the equations of motion in the HAMILTONian form:

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \qquad (i = 1, 2)$$

where $H = H(p_1, q_1, p_2, q_2)$. Consider p_1, q_1, p_2, q_2 as coördinates in space of four dimensions. In the neighborhood of a periodic solution the values p_1, q_1, p_2, q_2 corresponding to the different states of motion correspond to a three-dimensional torus, in consequence of the energy relation. As is well known, the problem can then be reduced to the HAMILTONian case $n = 1$, namely

$$\text{(1)} \qquad \frac{dp}{d\tau} = -\frac{\partial H}{\partial q}, \quad \frac{dq}{d\tau} = \frac{\partial H}{\partial p}$$

where $H = H(p, q, \tau)$, τ being an angular variable of period 2π which measures the distance of two points along the three-dimen-

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sional torus. The given periodic motion can be made to correspond to $p=q=0$.

We may represent this system in the following form:

$$\frac{dp}{d\tau}=P,\ \frac{dq}{d\tau}=Q,\ \frac{dr}{d\tau}=R=1 \tag{2}$$

where

$$P=-\frac{\partial H}{\partial q} \text{ and } Q=\frac{\partial H}{\partial p}$$

and r replaces τ in P and Q.

Let us consider p, q, r as rectangular coördinates in space of three dimensions.

The above equations give the direction of a stream line at every point of the (p, q, r)-space. The motions of the dynamical system are interpreted as the stream lines of a three-dimensional fluid in steady motion.

Consider the planes $r=0$ and $r=2\pi$; two points of these planes which have the same coördinates (p, q) are to be considered as congruent; they correspond to the same state of motion in consequence of the periodicity in τ.

Take a point P of the plane $r=0$ whose coördinates we denote by (p, q), and follow the stream line starting from P up to the point P_1 with coördinates (p_1, q_1) in which it meets the plane $r=2\pi$. The correspondence between the points P and P_1 furnishes a transformation T of the (p, q)-plane into itself. For this transformation the origin $p=q=0$ is an invariant point.

The transformation T has two important properties. In the first place the quantities P, Q, R satisfy to the relation

$$\frac{\partial P}{\partial p}+\frac{\partial Q}{\partial q}+\frac{\partial R}{\partial r}=0.$$

This means that the stream flow in space for which the velocity components are P, Q, R, is that of an incompressible fluid.

Secondly if we consider a small closed curve in the plane $r=0$ and the cylinder of height h bounded by the stream lines passing through the points of this curve and the planes $r=0$ and $r=h$, it is clear that after an interval of time 2π this cylinder goes into a like cylinder of height h with an equal area σ_1 of the corresponding curve in the plane $r=2\pi$ as base; this is a consequence of incompressibility. In other words the transformation T preserves areas. Consequently the Jacobian

$$J\begin{pmatrix} p_1, & q_1 \\ p, & q \end{pmatrix}$$

is 1.

Hence to the dynamical problem corresponds a certain area-preserving transformation T of the (p,q)-plane into itself with invariant point at the origin. To the important properties of the dynamical system for motions near to the given periodic motion correspond properties of T.

If H is analytic in p, q, τ then p_1, q_1 will also be analytic in p, q. Likewise if H is continuous together with all of its partial derivatives, the same will be true of p_1, q_1.

There arises now the interesting question as to whether or not there exists conversely a dynamical problem of this type for every such transformation T. The following result in this connection will be proved:

If

$$p_1 = \varphi(p,q), \quad q_1 = \psi(p,q)$$

is an area-preserving transformation T such that φ, ψ are continuous together with all of their partial derivatives, while the origin $p = q = 0$ is an invariant point, then there exists a corresponding dynamical system (1) *such that H is continuous together with all of its partial derivatives in p, q, τ and periodic of period 2π in τ.*

It would be of decided interest to establish a like result in the analytic case.

In the neighborhood of the origin the transformation T from (p,q) to (p_1, q_1) is essentially an affine transformation of determinant 1. Such a linear transformation can always be obtained by a one-to-one analytic deformation (rotation or stretching) which takes each point (p,q) into its transformed point $(\overline{p}_1, \overline{q}_1)$ while a parameter r varies from 0 to 2π. Upon this transformation may be superimposed the very small displacement with components

$$\frac{r}{2\pi}(p_1 - \overline{p}_1), \quad \frac{r}{2\pi}(q_1 - \overline{q}_1).$$

The combined transformation depending upon the parameter r leaves the origin invariant, yields a one-to-one transformation of the neighborhood of the origin into itself, and takes (p, q) into (p_1, q_1) as r increases form 0 to 2π.

As r varies from 0 to 2π, the points (p, q, r) describe arcs of curves joining $(p, q, 0)$ to $(p_1, q_1, 2\pi)$ in such a way that r increases, and the complete neighborhood of the r axis for $0 \leqq r \leqq 2\pi$ is filled by these in a one-to-one manner. If we set down all congruent arcs of curves obtained by a translation of space by a distance $2k\pi$ $(k = \pm 1, \pm 2, \ldots)$ in the direction of the positive r axis, all of (p, q, r) space in the neighborhood of the r axis is filled by curves made up of such arcs whose equations will have the form

$$p = f(r), \; q = g(r)$$

in which f and g are continuous together with all of their derivatives except for $r = 0, \pm 2\pi, \ldots$ where there may exist finite jumps in the derivatives.

Now imagine a deformation of the region $0 \leqq r \leqq 2\pi$ of (p, q, r) space in the direction of the r axis, in accordance with the formula

$$r = k \int_0^{\varrho} e^{\frac{1}{\varrho(\varrho - 2\pi)}} d\varrho$$

where the constant k is so selected that for $\varrho = 2\pi$, r is 2π also. Evidently r is thereby defined as a function of ϱ, continuous together with all of its derivatives for $0 \leqq \varrho \leqq 2\pi$, while all of these derivatives vanish for both $\varrho = 0$ and $\varrho = 2\pi$.

When this deformation of (p, q, r) space is made for $0 \leqq r \leqq 2\pi$, together with the corresponding congruent deformations of the regions

$$2k\pi \leqq r \leqq 2(k+1)\pi, \quad (k = \pm 1, \pm 2, \ldots),$$

a modified set of curves is obtained in which the functions f, g involved are everywhere continuous together with all of their derivatives.

Now let each point P move along its curve in the direction of the r axis with unit velocity. Any area σ in the plane $r = 0$ is thus carried into an area σ_0 in any plane $r = r_0$. But we have

$$\iint_{\sigma} dp\, dq = \iint_{\sigma_0} J\, dp\, dq$$

where $J(p, q, r)$ denotes the corresponding Jacobian. Consequently it is evident that the triple integral

$$\iiint J(p,q,r)\,dp\,dq\,dr$$

is invariant. Here J is not only continuous together with all of its derivatives, but is periodic in r of period 2π since $J(p,q,2\pi)=1$ by hypothesis.

Suppose now that we deform (p,q,r) space in the direction of q axis so that

$$\bar{q}=\int_0^q J(p,q,r)\,dq.$$

This deformation is evidently periodic in the desired sense and does not affect points in the planes $r=2k\pi$ $(k=0,\pm 1,\pm 2,\ldots)$. The new invariant integral is then simply the ordinary volume integral $\iiint dp\,dq\,dr$ in the modified variables.

The corresponding differential equations are

$$\frac{dp}{d\tau}=P,\quad \frac{dq}{d\tau}=Q,\quad \frac{dr}{d\tau}=1$$

where P, Q are continuous functions of p,q,r, together with their partial derivatives of all orders, and periodic of period 2π in r. Since volumes are invariant we have of course

$$\frac{\partial P}{\partial p}+\frac{\partial Q}{\partial q}=0.$$

This means that a function H of the same type exists for which

$$P=-\frac{\partial H}{\partial q},\quad Q=\frac{\partial H}{\partial p}.$$

In other words the given area-preserving transformation may be associated with a dynamical problem of the stated type.

This remark shows that the area-preserving property of the transformation used by Poincaré in his last geometric theorem is really its characteristic property. The remark shows also how the dynamical problem leads to the consideration of a transformation near an invariant point, or near a single closed invariant curve into which such a point may be expanded, rather than to a transformation defined over a complete ring as required by Poincaré. It was for this reason that I developed a modification of Poincaré's theorem for a transformation of this less restricted type, which seems more appropriate to many of the actual dynamical applications. Indeed the more detailed consideration of these applications shows that for many purposes the use of Poincaré's last geometric theorem in a modified self-evident form will suffice, once the analytic details are developed.[1])

From a general topological point of view the plane translation theorem of BROUWER may be looked upon as dealing with the morphology of continuous one-to-one transformations of the sphere with only a single invariant point. In the same way a suitable extension of the theorem of POINCARÉ[2]) throws light on the morphology of any such transformation of the sphere with only two such points. An important new method of attack devised by KERÉKJÁRTÓ but not yet published[3]) seems to afford a means of treating these and other similar questions on a common fundamental basis.

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[1]) Cf. chapter 6 of my book on *Dynamical Systems*, New-York, 1927.

[2]) See my paper: An extension of POINCARÉ's last geometric theorem, *Acta Mathematica* **47** (1925), p. 297.

[3]) See p. 86 of this volume (note of the editors).