The plane translation theorem of Brouwer and the last geometric theorem of Poincaré.

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Researches on topological transformations of surfaces lead to the following fundamental questions. One of them concerns the determination of the different classes of invariant points and their indices; to this category belong the theorems of Brouwer, Birkhoff, Alexander and Nielsen. We refer to the excellent work of J. Nielsen[1]) which contains a general and systematic treatment of this kind of problem.

The other problem is to investigate the structure of surface transformations that is to say to study their *morphology* (using a very appropriate expression of Prof. Birkhoff). To this sort of question belong the plane translation theorem of Brouwer, the last geometric theorem of Poincaré, its extensions given by Birkhoff, the general translation theorem of Brouwer (which is yet unproved) and the results of Birkhoff on iterations of topological transformations.

The main problem for questions in this category is to determine a *transformation-field* that is a *maximal region lying outside its image:* this general notion is due to the profound investigations of Brouwer.[2])

The method of Brouwer for constructing a transformation-field is based on the analysis situs of general plane continua;[2]) this way

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1) J. Nielsen, Untersuchungen zur Topologie der geschlossenen zweiseitigen Flächen, *Acta Mathematica,* 50 (1927), p. 189—358.

2) L. E. J. Brouwer, Über eineindeutige stetige Transformationen von Flächen in sich, *Math. Annalen,* 69 (1910), p. 176—180. See his references there to the *Proc. of the Royal Academy at Amsterdam.*

is difficult and as BROUWER has admitted the conclusions are unsufficient. The proof of the plane translation theorem given by BROUWER[3]) is complete but too complicated. Recently SCHERRER[4]) has outlined a new method which seems to be more simple than that of BROUWER but which is restricted to the plane translation theorem.

The general proof of POINCARÉ's last geometric theorem given by BIRKHOFF[5]) is extremely ingenious; but it seems to me that it is especially related to this single theorem and cannot be employed to similar problems of more general type.

The aim of the present paper is to give a general and elementary method of constructing a transformation-field which might be employed for several questions of morphology of surface transformations. In the first two paragraphs the general method is developed; in the third and in the fourth paragraph I employ the general method to prove the plane translation theorem of BROUWER and the last geometric theorem of POINCARÉ. — It may have some interest to observe that my method of proving the theorem of POINCARÉ is very similar to the way in which POINCARÉ has tried to prove it; in essence it may be considered as the completion of his attempts.[6])

The further development and other applications of the same method will be given in a subsequent paper of the author. Among others there will be treated the general theorem of BROUWER concerning translations on arbitrary surfaces and the other abovementioned topics of morphology.

1. Preliminaries.

1. 1 Lemma on arcus-variation.

Lemma. Let j and j' be two simple closed curves in the plane, which have an arc α in common such that the interiors of j and j' both lie on the same side of α. Let t be a topological

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[3]) L. E. J. BROUWER, Beweis des ebenen Translationssatzes, *Math. Annalen*, **72** (1912), p. 36—54.

[4]) W. SCHERRER, Translationen über einfach zusammenhängende Gebiete, *Vierteljahrsschr. d. Naturforsch. Ges. Zürich*, **70** (1925), p. 77-84.

[5]) G. D. BIRKHOFF, An extension of POINCARÉ's last geometric theorem, *Acta Mathematica*, **47** (1925), p. 297—311.

[6]) H. POINCARÉ, Sur un théorème de géométrie, *Rendiconti d. Circ. Mat. di Palermo*, **33** (1912), p. 375—407.

(i. e. continuous one-to-one) transformation of j into j' which preserves the sense of orientation and has no invariant points. If there exists an arc β of $j-\alpha$ and an arc γ of $j'-\alpha$, each containing all common points of $j-\alpha$ and of $j'-\alpha$ and such that *1)* the image β' of β has no point in common with γ except extremities, *2)* the arcs α and β' do not separate the arcs α' and γ from one another on j' (where α' and β' denote the images of α and β respectively) then *the variation of argument of the transformation-vector in a positive circuit of the curve j is equal to 2π.* (Fig. 1).

We may deform the curve j' by little so that the arcs α and γ have no extremities in common and that all the common points of $j-\alpha$ and $j'-\alpha$ are internal points of γ. Replace the arc γ by an arc γ_1 which does not meet j, except for one point of γ at most, in such a way that the arcs $j'-\gamma$ and γ_1 form a simple closed curve j'' whose interior lies on the same side of α as the interior of j'. Define a topological transformation of j into j'', say t', such that the image of γ by the inverse of t is transformed under the transformation t' into γ_1, while otherwise the transformations t and t' are the same. In consequence of our assumption *1)* the arc γ^* which is carried under t into the arc γ has no point in common with $\beta+\gamma$; furthermore the arcs γ^* and α belong to the arc $j-\beta$, and the interiors of j'' and j' both lie on the same side of α. Consequently the angular variation of a vector whose origin describes the arc γ^* and whose extremity describes first the arc γ as image of the origin under t, and then the arc γ_1 as image of the origin under t' is in both cases the same.

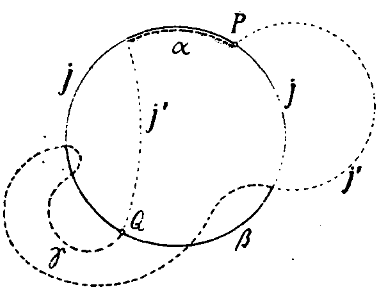

Fig. 1.

If the whole arc $j''-\alpha$ lies inside (or outside) the curve j, then one of the curves j and j'', say j'', is contained in the interior of the other one, save for the common points, for the interiors of j and j'' lie on the same side of α; we deform j'' continuously into a single point in its interior; the total variation of the transformation-vector changes continuously, moreover it is a multiple of 2π, so that it is just 2π.

If neither of the curves j, j'' lies inside the other, then $j''-\alpha$

consists of two arcs, one of them being interior, the other one exterior to the curve j; let Q be their common extremity on j and let P be the common extremity of α and of the arc of $j''-\alpha$ lying outside j; denote by P', Q' the images of P and Q. From the condition *2)* of the lemma it follows that the pairs of points P', Q and P, Q' do not separate one another on j''. By a continuous deformation of the transformation t' of j into j'', under which no point of j coincides with the corresponding point of j'', we carry the point P' through Q into a point of j'' lying inside j; in the same way we carry the point Q' through P into a point of j'' inside j. To the arc QP of j which lies inside j'' corresponds then the arc $Q'P'$ of j'' lying inside j. Deform the curve j'' in its interior into a single point lying inside j, in such a way that the arc $Q'P'$ remains continually inside j. Under this deformation no point of j coincides with the corresponding point of j''; hence the total variation of the transformation-vector changes continuously and for the final position it is equal to 2π. — This completes the proof of the above lemma.

1. 2 On translation-arcs.

Let t denote a sense-preserving topological transformation of the plane into itself without invariant point.

We und rstand by a *translation-arc* a simple continuous arc AB such that B is the image of A under the given transformation t and that the image of the arc AB does not contain any interior point of the arc AB.

A translation-arc passing through any given point of the plane can be constructed in the following simple way due to BROUWER. Join P to its image P' by the straight segment PP'; let a variable point starting from P describe this segment up to the first point for which the described path and its image have a common point Q'. If Q' coincides with P' then the segment PP' is a translation-arc. If Q' is different from P', denote by Q'' and Q the image of Q' by the direct and inverse transformations respectively. To fix our ideas, suppose that at the point Q' the segment PQ' abuts on a side of its image, that is the arc $P'Q''$. (If the arc $P'Q''$ abuts on a side of the segment PQ', we interchange in the following consideration the rôle of P with that of P' and the rôle of the transformation t with that of its inverse). We choose a point B on PQ' very near to Q' and denote by A and C its inverse and direct images respectively. We join A to P by a continuous arc AP which remains very near to PQ' and lies on one side of it; under such circumstances the arc AP does not intersect its image. Consequently the arc composed of the arc AP and of the segment PB forms a translation-arc passing through the point P.

Theorem I. *A translation-arc AB and its images BC and CD under the transformations t and t^2 together form a simple arc.*[7])

Suppose *first* that $AB+BC$ is not a simple arc; then A coincides with C or with an internal point of the arc BC. Denote by j the simple closed curve composed of the arc $\beta = AB$ and of the subarc $\alpha = BA$ of BC. The image of j is a simple closed curve j' composed of the arc BC and of the subarc CB of CD. Denote the subarc CB of CD by γ and observe that γ is identical with the image α' of α. The arc α is common to the curves j and j' and their interiors lie on the same side of it. The image β' of β, that is the arc BC, has no point in common with the arc $\gamma = CB$ except extremities. Thus all the conditions of the Lemma 1.1 are realized; hence it follows that the transformation t has an invariant point inside j. But this is contradictory to our assumption about the transformation.

Secondly we assume that the *simple* arc $AB+BC$ has a point in common with the arc CD; this common point cannot belong to BC. Describe the arc CD starting from C and denote by P its first point belonging to AB. Denote by β the subarc PB of AB and by j the simple closed curve which is composed of the arcs β, BC and the subarc CP of CD. Let DE denote the arc which is the image of CD, and P' the image of the point P. The image of j is the simple closed curve j' composed of the subarc $P'C$ of BC, the subarc DP' of DE and the arc CD. Call α the arc composed of the subarc $P'C$ of BC and the subarc CP of CD. The curves j and j' have the arc α in common and their interiors lie on the same side of it. Let γ denote the arc composed of the subarc PD of CD and the subarc DP' of DE. Then we recognize that all conditions of Lemma 1.1 are satisfied; in fact the image β' of β is the arc $P'C$ which has no internal

[7]) Cf. Brouwer, *Math. Annalen*, **72** (1912) p. 38—39.

point in common with γ; the arcs α' and γ have the arc DP' in common, so that the condition *2)* also is fulfilled. Thus a contradiction would arise as above.

1. 3 Arc abutting on its image.

We represent the arcs AB and BC (and also CD) as straight segments of unit length on the x axis so that B is the origin of coordinates and lies to the right of A. Then we mark the *upper* and *lower side of the segment* AC and the *right-hand-side and left-hand-side of an arc starting from a point of* AC. We observe that in consequence of the assumption that the transformation t preserves sense, the upper and the lower side of the segment AB is carried by the transformation into the upper and the lower side of BC respectively, and the right- and left-hand-sides of an arc starting from B go over respectively into the right- and left-hand-sides of the image arc starting from C. (The strict meaning of these intuitive notions is assured by JORDAN's theorem on simple closed curves).

Consider a simple arc, starting from the point B, which does not meet the segment AC elsewhere. Describe this arc up to the first point for which the described path b and its image have a common point. We shall say that *the arc b abuts on its (direct or inverse) image* and we understand by this expression that the arc b and its image have a common point, while no proper subarc of b of origin B meets its image. Let a and c denote the inverse and the direct image of b respectively.

b and c have just one common point; either the extremity P of b is an internal point of c or, vice versa, the extremity P' of c is an internal point of b. If P belonged to c and at the same time P' to b, then the arc PbP' of b would be a translation-arc, for its internal points do not belong to c, and P' is the image of P; the image of PbP' would then be a proper subarc of $P'cP$, in consequence of Theorem I. By interchanging the rôle of PbP' and $P'cP$ and considering t^{-1} instead of t, the opposite of the above statement would result. From this contradiction we conclude that b and c have just one point in common. The same holds true concerning the arcs a and b.

Theorem II. *Let b be a simple arc of origin B which does not meet the segment AC elsewhere and which abuts on its image.*

The direct and inverse images c and a of b cannot meet the segments AB and BC respectively.

Suppose the arc a meets BC; describe the arc a starting from A and denote by Q its first point belonging to BC (Fig. 2). Call j the simple closed curve composed of the arc AaQ and of the segment ABQ. Let Q' be the image of Q; the image of j is the simple closed curve j' composed of the arc BbQ' and of the segment BCQ'. The segment $\alpha = BQ$ belongs to j and to j'; as the interior of j

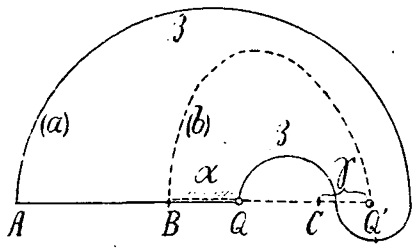

Fig. 2

corresponds by the transformation to the interior of j' and the upper side of ABQ goes over into the upper side of BCQ' it follows that the interiors of j and j' both lie on the same side of α. Denote by β the arc AaQ and by γ the segment CQ'. Both β and γ contain all the common points of $j - \alpha$ and of $j' - \alpha$. The image β' of β is the arc BbQ' which has no point on γ except its extremity Q'. Finally we have $\alpha' = \gamma$ so that condition *2)* of the Lemma 1. 1 is also satisfied. From this would follow the existence of an invariant point inside j, contrary to our assumption.

From the above proof we obtain immediately the following

Theorem II'. *Let QS be a translation-arc, and ST its image. Let KL be a subarc of QST which contains the image K' of K. If λ denotes a simple arc which forms together with the arc KL a simple closed curve, then λ must intersect its own image.*[8])

We wish to prove the following

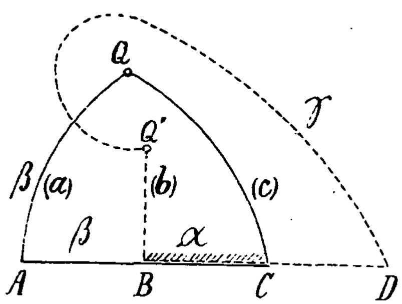

Fig. 3.

Theorem III. *The arcs a and c have no point in common.*

Suppose the contrary and denote by Q the first point of a counted from A which belongs to c (Fig. 3.); the arcs AaQ, CcQ and the segment ABC together form a simple closed curve j (cf. Theorem II). Either AaQ must be a proper subarc of a, or CcQ

[8]) Cf. Brouwer, *Math. Annalen* **72** (1912), p. 44., Satz 6.

a proper subarc of c; otherwise the common extremity Q of a and c would be transformed by t^2 into itself, which is excluded by Theorem I. Suppose then AaQ to be a subarc of a; its image is a subarc BbQ' of b which does not meet either of AaQ and CcQ. The image DQ' of CcQ cannot meet CcQ except in the point Q. Now let α be the segment BC, which is a common arc of the curve j and of its image j'; the interiors of j and j' both lie on the same side of α for the same reason as in the proof of Theorem II. Denote by β the arc composed of the arc AaQ and the segment AB: and by γ the arc composed of CD and DQ'. Obviously all the conditions of the Lemma 1·1 are satisfied including the condition *2)* also, for the image α' of α, that is the segment CD, belongs to γ. From this a contradiction arises.

Now we want to show that the bounded region determined by the arcs a, b, and the segment AB, which we denote by (a, b, AB) abuts on the upper or lower side of AB according to whether b goes from B upwards or downwards. Otherwise the segment BC which has no point on a, b, AB except B, would lie in the region (a, b, AB). In consequence of Theorems II and III the arc c would then lie in the same region. The image of the region (a, b, AB), that is the region (b, c, BC), would abut on the upper side of BC while the region (a, b, AB) abuts on the lower side of AB. This is a contradiction as has been observed above.

Finally we observe that a cannot abut on the right-hand-side of b. (In the same way b cannot abut on the left-hand-side of a.) Otherwise the extremity of b would be separated from the point C by the simple closed curve composed of AB, a and a subarc of b; but the arc c joins the point C to the extremity of b. This is a contradiction to the Theorems II and III.

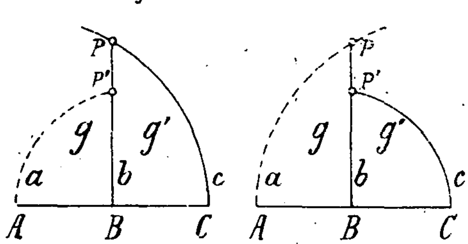

Fig. 4.

Our result may be summarized in the following

T h e o r e m IV. *Let b be a simple arc starting upwards from B and not meeting the segment AC elsewhere; suppose that b abuts on its image; denote by a and c its inverse and direct image respectively. Either a abuts on the left-hand-side of b and b abuts on* *the left-hand-side of c; or else c and b abut on the right-hand-side of b and a respectively. The region g determined by (a, b, AB) is transformed into the region g' determined by (b, c, BC). Both g and g' abut on the upper side of ABC; these two regions have no interior point in common.* (cf. Fig. 4.)

Let P denote the extremity of b and P' the extremity of a or of c, which lies on the arc b; the point P' is the image of P by the inverse or direct transformation. The arc BbP' belongs to the boundaries of both of the regions g and g'. The arc PbP' belongs to the boundary of just one of these regions while it lies in the exterior of the other region. This arc PbP' will be called the *free arc* of b; it is obviously a *translation-arc.* That side of the free arc which is turned towards the exterior of the regions g, g' will be called its *free side.*

2. Method of construction of a transformation-field.

2. 1 The quantities ε and η.

Denote in general by R_n the square $-n \leqq x \leqq n, \; -n \leqq y \leqq n$ $(n = 1, 2, \ldots)$. Let ε_n be the minimal distance of an arbitrary point of R_n from its direct or inverse image. Denote by η_n the largest number between 0 and $\varepsilon_n/2$ such that two arbitrary points of R_n whose distance is less than or equal to η_n are carried by the direct (or inverse) transformation into two points whose distance is less than or equal to $\varepsilon_n/2$.

2. 2 Determination of a base-point on the free segment.

Apply Theorem IV to the case where b is a straight segment perpendicular to the segment AC. Let a variable point describe the perpendicular to AC starting upwards from the point B to the first point for which the described path b and its image have a common point; denote by a and c the inverse and the direct image of b respectively. Let P' denote the extremity of a or of c which lies on b. Suppose, to fix our ideas, P' to be the direct image of P. Denote by $P'P''$ and by $P^{-1}P$ the arcs into which the segment PP' is carried by the direct and by the inverse transformation, respectively.

Let R_n be the smallest square (n integer) which contains P and P'; ε_n and η_n signify the quantities defined in 2. 1. We understand by a *mid-segment* of PP' that segment of PP' whose extremities are distant η_n from P and P' respectively.

We understand by a *base-point* B_1 a point of the mid-segment of PP' such that there exists a segment perpendicular to PP', starting from B_1 towards the free side of PP', which meets its own image and does not meet the direct or inverse image of PP'.

The *existence of a base-point* can be shown by the following consideration. Let S be an arbitrary point of the mid-segment of PP'; take the perpendicular to PP' starting from the point S towards the free side of PP' and denote by S_1 its first point belonging to $P^{-1}P+P'P''$. Suppose that contrary to our statement, no segment SS_1 meets its image. An immediate application of Theorem II', having regard to the determination of η_n, leads to the result that if S lies within η_n of P or P' the corresponding point S_1 belongs in the first case to the arc $P^{-1}P$, in the second case to the arc $P'P''$. Those points S of the mid-segment for which the point S_1 belongs to the arcs $P^{-1}P$, and $P'P''$ form two intervals; let S^* be their common extremity (Fig. 5.). Denote by S_1^* and S_2^* the first points of the perpendicular to PP' starting from S^* towards the free side of PP', which belong to $P'P''$ and $P^{-1}P$ respectively; suppose for instance that S_1^* lies between S^* and S_2^*. We have assumed that the segment $S^*S_1^*$ does not meet its image; in consequence of Theorem II' the direct image T^* of S^* must then lie on the subarc S_1^*P'' of the arc $P'P''$. According to whether the inverse image Q^* of S^* belongs to the subarc PS_2^* or $S_2^*P^{-1}$ of PP^{-1} apply Theorem II' to the segment $\lambda=S^*S_2^*$ and $\lambda=S_1^*S_2^*$ respectively; hence we obtain that the segment $S^*S_2^*$ intersects its image. The same holds valid of a segment SS_1 whose origin S is a point on PS^* near the point S^*. From this contradiction of our assumption the existence of a base-point follows as stated above.

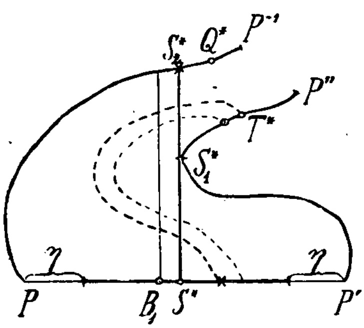

Fig. 5.

The *construction is continued in the following way.* We determine a base-point B_1 on the free segment PP' and take the perpendicular to PP' starting from B_1 towards the free side, as far as the first point P_1 for which the segment B_1P_1 meets its image; let P_1' be the (direct or inverse) image of P_1 lying on the segment B_1P_1; we determine on the free segment P_1P_1' a base-point B_2 and take the perpendicular to P_1P_1' starting from B_2 towards the free side of P_1P_1'; and so on.

At every stage of this process the broken line $BB_1B_2 \dots B_kP_k$ is to be considered as an arc b abutting on its image and thus Theorem IV may be applied.

Another important feature of this process is that *the continuation of the construction is uniquely determined by the last free segment P_kP_k' (and the base point B_{k+1}) and thus does not depend on the preceding part of the polygonal line $BB_1B_2 \dots$*

2. 3 Deviation of the path.

Let l be an infinite straight line parallel to the x or y axis. We understand by *deviation of the path by the line l* the following modification of the above construction.

Retaining the notations of 2. 2, take the perpendicular to PP' starting from B_1 towards the free side of PP'; let M be the first point of this line on l. If the segment B_1M does not meet its image, we proceed on l from M in either of the two directions up to the first point P_1 (or $\overline{P}_1$) such that the broken line B_1MP_1 (or $B_1M\overline{P}_1$) meets its image.

Applying Theorem IV to this case we determine the free arc P_1P_1' (or $\overline{P}_1\overline{P}_1'$) on the broken line B_1MP_1 (or $B_1M\overline{P}_1$). If one of these free arcs consists of a single straight segment we can determine on this a base-point accordingly to 2. 2 and carry on the construction. The same is valid if the free arc P_1P_1' (or $\overline{P}_1\overline{P}_1'$) is composed of two straight segments provided that he free side corresponds to the angle $\pi/2$. If each of the free arcs P_1P_1' and $\overline{P}_1\overline{P}_1'$ consists of two segments and the angle corresponding to the free side is $3\pi/2$ for both of them, then the line B_1MP_1 abuts on its direct, the line $B_1M\overline{P}_1$ on its inverse image (or vice versa) (Fig. 6.). Describe the segment MP_1 from M up to the first point T which belongs to the direct or inverse image of $M\overline{P}_1$; let S be the inverse or direct image of T, respectively. The segment ST is then a translation-arc; on this a base-point B_2 can be determined according to the result of 2. 2. From the base-point

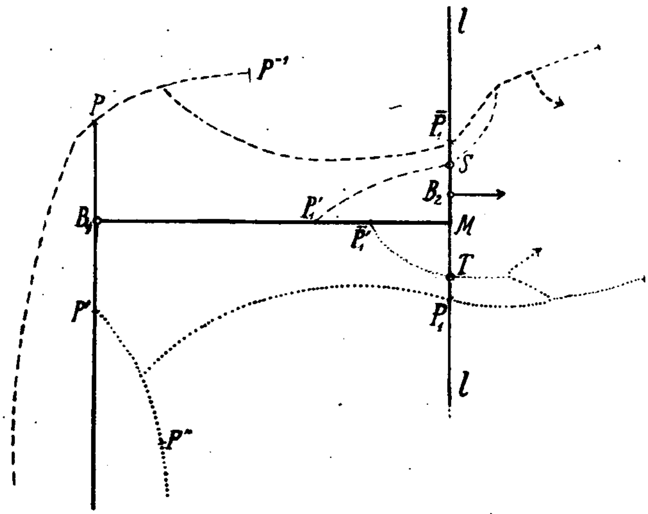

Fig. 6.

B_2 we carry on the construction in conformity with the above description.

We would emphasize that *if after deviation the line* $BB_1B_2\ldots$ *crosses the line* l *(that is to say if* B_1 *and* B_3 *lie on opposite sides of* l*) then the corresponding base-point* B_2 *belongs to a translation-arc* ST *of the line* l.

3. The plane translation theorem of Brouwer.

Let t be a topological transformation of plane into itself which preserves the sense of orientation and has no invariant point. We understand by a *translation-field* a region lying outside its image, and bounded by two simple open lines such that one of these lines is the image of the other one by the given transformation. By a *simple open line* is meant a topological image of the infinite straight line such that points of the straight line converging to infinity correspond to a similar sequence of points on the simple open line.

The *theorem of* BROUWER *on plane translations* can be stated as follows:

If a sense-preserving topological transformation of the plane into itself has no invariant point, a translation-field can be constructed which contains any given point of the plane.

Construct a translation-arc AB through the given point (cf. 1. 2) and determine a system of rectangular coordinates x, y in such a way that AB and its image BC on the x axis are represented as segments of unit length. B is the origin of coordinates and C is to the right of B.

Take the perpendicular to AC starting upwards from B and describe this up to the first point P for which the segment BP and its image meet. Let P' be the (direct or inverse) image of P belonging to the segment BP. On the free segment PP' we choose a base-point B_1 (cf. 2. 2) and take the perpendicular to PP' starting from B_1 towards the free side of PP'; and so on.

If at any stage of this process a base-point B_k lies outside the square R_{n+1} and inside R_{n+2}, furthermore if the perpendicular to $P_{k-1}P'_{k-1}$ starting from B_k abuts on a side, say $y=n$, of the square R_n before meeting its own image, then we proceed on the line $y=n$ in either of the two directions and determine the next base-point B_{k+1} accordingly to 2. 3.

The polygonal line $b=BB_1B_2\ldots$ obtained by indefinite continuation of this construction cannot return to a point of the segment ABC; otherwise a part of the line b and a segment of ABC together would form a simple closed polygon whose image would lie in its interior; inside this polygon the transformation would then have an invariant point.

Denote by a and c the inverse and the direct images of b respectively. From Theorem IV it follows that no two of the lines a, b, c have a common point; hence we also conclude that the line b cannot return to one of its anterior points.

We carry out the same construction starting from B towards the lower side of ABC and again denote the line $\ldots B_{-2}B_{-1}B\,B_1B_2\ldots$ by b, its images by a and c. The same considerations lead to the result that no two of the lines a, b, c have a common point.

In order to prove that b is a simple open line, let us first observe that any square R_{n+1} contains only a finite number of the base-points of the construction. For consider the complex of lines: $P_kP'_k+B_{k+1}B_{k+2}$. If $l\geqq k+2$, the complexes $P_kP'_k+B_{k+1}B_{k+2}$ and $P_lP'_l+B_{l+1}B_{l+2}$ have no point in common, in consequence of the construction. Every free segment (or free arc) which lies in R_{n+1} has a diameter $\geqq\varepsilon_{n+1}$; the base-point B_{k+1} is at a distance $\geqq\eta_{n+1}$ from the points P_k and P'_k, so that the segment B_kB_{k+1} is of diameter $\geqq \eta_{n+1}$ for it contains the point P_{k+1}. The segment $B_{k+1}B_{k+2}$ contains then a straight segment of diameter $\geqq \eta_{n+1}$. The whole complex $P_kP_k' + B_{k+1}B_{k+2}$ consists essentially of three segments of length $\geqq \eta_{n+1}$ which are parallel to the axes. Hence it follows that there is only a finite number of base-points in the square R_{n+1}. Now a segment B_kB_{k+1} of the line b whose origin B_k lies outside of R_{n+1} cannot enter into R_n; consequently there is only a finite number of segments B_kB_{k+1} of the line b which have points inside of the square R_n. This means that a set of points lying on different segments of the line b cannot have any limiting point at finite distance, that is to say that the line *b is a simple open line.*

The lines a and b bound a translation-field whose image is the region bounded by the lines b and c; the two regions lie on different sides of the line b and they have no point in common.

This completes the proof of the plane translation theorem of BROUWER.

4. The last geometric theorem of Poincaré.

The theorem of POINCARÉ can be stated in the following form:

Consider a topological transformation of a ring-shaped plane surface into itself which displaces the points on the inner boundary in the negative, the points on the outer boundary in the positive sense. Either the transformation leaves at least one point invariant, or else there is a simple closed curve separating the two boundaries which lies completely inside its (direct or inverse) image.

Let us suppose that the transformation has no invariant point and construct a curve lying inside its image.

Let (r, φ) denote polar coordinates in the ring-shaped surface: $1 \leqq r \leqq 2$, and let the transformation be given by the formulae: $r' = R(r, \varphi)$, $\varphi' = \theta(r, \varphi)$. Determine the value of $\theta(r, \varphi)$ for points of the outer boundary $r = 2$ in such a way that $\varphi > \theta(2, \varphi)$; then for the inner boundary $\theta(1, \varphi) < \varphi$; this is the expression of the condition that the transformation turns the two boundaries in opposite senses.

We map the ring on to the strip: $1 \leqq y \leqq 2$, $-\infty < x < +\infty$ by means of the formulae: $y = r$, $x = -\varphi + 2k\pi$ $(k = 0, \pm 1, \pm 2, \ldots)$, where (x, y) denote rectangular coordinates in the plane. To the given transformation of the ring corresponds a topological transformation of the strip into itself with no invariant point, which is periodic in x, of period 2π. Denote by (x', y') the image of the point (x, y) then the relations exist: for $y=1: x'>x$ and for $y=2: x'<x$.

Denote by B the point $(x, y)=(0, 1)$. Starting from the point B describe the segment $x=0, 1\leq y\leq 2$ up to the first point P for which the segment BP and its image meet. Denote by P' the direct or inverse image of P which lies on the segment BP. (cf. Theorem IV).[9]) Determine a base-point B_1 on the free segment PP', take the perpendicular to PP' starting from B_1 towards the free side and describe this up to the first point P_1 for which the segment B_1P_1 and its image meet. Let P_1' be the direct or inverse image of P_1 lying on the segment B_1P_1; on the free segment P_1P_1' determine a base-point B_2, take the perpendicular to P_1P_1' starting from B_2 towards the free side; and so on.

Denote by l_k the vertical segment $1\leq y\leq 2, x=2k\pi$ $(k=0, \pm 1, \pm 2, \ldots)$ and by l any one of them. Let l^1 and l^{-1} be the direct and the inverse images of l respectively. We may assume that *the arcs l and l^{-1} have only a finite number of common points*;[10]) denote them by $H_1, H_2, \ldots, H_n$ and their images by $H_1', H_2', \ldots, H_n'$; the H'-s are the common points of l^1 and l. Consider the segments H_iH_i'; if one of them, say H_jH_j' contains another segment H_iH_i' then we omit H_jH_j'; those which remain will be denoted by $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)}$. (Such segments lying on l_k will be denoted by $\lambda_k^{(1)}, \lambda_k^{(2)}, \ldots, \lambda_k^{(r)}$.) Each segment $\lambda^{(i)}$ is a *translation-arc*, for its extremities correspond to one another by the transformation and the segment does not contain any other pair of points corresponding to each other by the transformation. The $\lambda^{(i)}$'s are the only translation-arcs on the segment l.

An initial part of the line $b=BB_1B_2\ldots$ constructed by the above process is contained in the rectangle between l_0 and l_1 (or between l_{-1} and l_0; suppose however the first, to fix the ideas). *The line b cannot abut on the lines $y=1$ or $y=2$*; the first part

[9]) From the preliminary considerations of paragraph 1. we need for Poincaré's theorem only the lemma 1. 1 in a weaker form and as its direct consequence, the theorems III and IV.

[10]) Otherwise we subdivide l into a finite number of sufficiently small segments and replace every one of them by an approximating arc of the same extremities which meets l, l^1 and l^{-1} in a finite number of points (cf. Kerékjártó, Vorlesungen über Topologie, I., Berlin 1923, p. 89. Satz II.). We consider then instead of l the line obtained by this modification.

of this statement is an obvious consequence of the construction; the second part follows from the condition that the points on $y=1$ and on $y=2$ are displaced in the opposite senses, so that the line b whose origin lies on the line $y=1$ and whose extremity would lie on the line $y=2$ would intersect its image.

An essential feature of the construction is that any finite part of the plane contains only a finite number of the base-points of the construction (cf. § 3.). Consequently the line b must leave the rectangle between l_0 and l_1 at some stage of the construction. Let M be the first point of b on the segment l $(=l_0, l_1)$. We *deviate the path by the segment l,* that is to say we proceed from the point M along the segment l upwards or downwards as far as the first point for which the path and its image meet. From the result of 2. 3 it follows that if after deviation the line b crosses the line l, the corresponding base-point lies on a translation-arc of l, that is to say on a segment $\lambda^{(i)}$.

We carry on the construction of the line b in the same way and deviate it by each segment l on which it abuts. In this way we obtain a line b such that b crosses any one of the segments l only on the translation-arcs $\lambda^{(i)}$. (Fig. 7.)

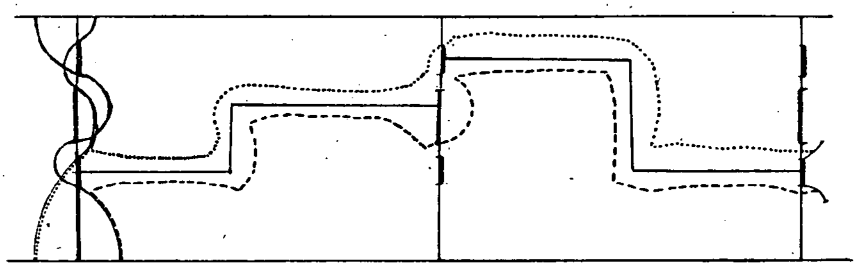

Fig. 7.

After sufficiently long continuation of the construction the line b must leave the rectangle between l_{-r} and l_r (r denotes the number of the translation-arcs $\lambda^{(i)}$ on l). Consequently there are two segments l_m and l_n which are crossed by b on congruent translation-arcs $\lambda_m^{(i)}$ and $\lambda_n^{(i)}$ (of the same superscript i). Suppose that b meets the translation-arc $\lambda_m^{(i)}$ before the translation-arc $\lambda_n^{(i)}$. We keep only that part of the line b which ends at $\lambda_n^{(i)}$. Now we choose on $\lambda_n^{(i)}$ the same base-point as on $\lambda_m^{(i)}$ and carry on the construction in a periodic way (of period $2\pi\omega$, where ω denotes the integer $m-n\neq 0$); this is possible in consequence of the periodicity of the transformation in x and of the characteristic of the process that the continuation is determined only by the last free segment (cf. 2. 2).

The line b obtained in this way does not intersect its image; b is periodic in x, of period $2\pi\omega$, apart from a certain initial part of b. Omit this initial part of b and from the remaining part construct by the translations $x'=x+2k\pi\omega$, $y'=y\,(k=0, \pm 1, \pm 2, \ldots)$ an open line W which is totally periodic in x, of period $2\pi\omega$. Then the line W does not intersect its image; in other words the line W separates its direct or inverse image from the line $y=1$. The same holds true of the lines $W_0, W_1, W_2, \ldots, W_{\omega-1}$ obtained from $W=W_0$ by the translations $x'=x+2k\pi, y'=y, (k=0, 1, \ldots, \omega-1)$. The lines $W_0, W_1, \ldots, W_{\omega-1}$ together determine a region abutting on the line $y=1$ whose other boundary is formed by a *simple open line W^* periodic in x, of period 2π*. The line W^* separates its direct or inverse image from the line $y=1$.

To the line W^* corresponds in the ring-shaped surface a simple closed curve which separates the two boundaries from one another and which lies in the interior of its direct or inverse image.

Thus the theorem of POINCARÉ is proved. I called attention in my „Vorlesungen über Topologie," I., p. 210. to the possibility of proving POINCARÉ's theorem by an adequate construction of a translation-field.

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The extension of the theorem given by BIRKHOFF concerning transformations which take the ring into another ring with the same inner boundary, can be proved in the same way. The profound result of BIRKHOFF on the existence of *two distinct invariant points* can be deduced by an accurate investigation of the structure of the transformation near an invariant point. I propose to return to these questions in a subsequent communication.

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