On a generalisation of a theorem of Kakeya.

By E. EGERVÁRY in Budapest.

1. (A) It is generally known, that the moduli of the roots of an algebraic equation

$$c_0+c_1z+\ldots+c_nz^n=0 \tag{1}$$

are not larger than the only positive root of the equation

$$|c_0|+|c_1|z+\ldots+|c_{n-1}|z^{n-1}-|c_n|z^n=0. \tag{2}$$

(B) An immediate extension of this theorem is the following theorem of A. E. PELLET:

If the equation

$$|c_0|+|c_1|z+\ldots+|c_{\nu-1}|z^{\nu-1}-|c_\nu|z^\nu+|c_{\nu+1}|z^{\nu+1}+\ldots+|c_n|z^n=0 \tag{3}$$

has two positive roots, r_1 *and* $r_2\geqq r_1$, *then the equation* (1) *has* ν *roots whose moduli are not larger than* r_1, *and* $n-\nu$ *roots whose moduli are not smaller than* r_2.[1])

[1]) A. E. PELLET, Sur un mode de séparation des racines des équations et la formule de LAGRANGE, *Bulletin des sciences math.*, (2) **5** (1881), pp. 393–395. See also T. RADÓ, Algebrai egyenletek gyökeiről, Math. és Phys. Lapok, **28** (1921), pp. 30–37. The theorem of PELLET can be reduced very simply to the theorem (A) mentioned above. The equation (3) has two distinct positive roots, r_1 and r_2, if

$$|c_\nu|>\underset{0\leqq z\leqq\infty}{\mathrm{Min}}\left(\frac{|c_0|}{z^\nu}+\ldots+\frac{|c_{\nu-1}|}{z}+|c_{\nu+1}|z+\ldots+|c_n|z^{n-\nu}\right).$$

If now $r_1<|z|<r_2$, then

$$\left|\frac{c_0}{z^\nu}+\ldots+\frac{c_{\nu-1}}{z}+c_{\nu+1}z+\ldots+c_nz^{n-\nu}\right|\leqq$$

$$\leqq\frac{|c_0|}{|z|^\nu}+\ldots+\frac{|c_{\nu-1}|}{|z|}+|c_{\nu+1}|\,|z|+\ldots+|c_n|\,|z|^{n-\nu}<|c_\nu|,$$

from which it follows, that (1) has no root in the region $r_1<|z|<r_2$.

Hence, if we put

2. If the coefficients of an algebraic equation

$$a_0 + a_1 z + \ldots + a_n z^n = 0 \tag{4}$$

satisfy the inequalities

$$0 < a_0 \leqq a_1 \leqq \ldots \leqq a_n, \tag{5}$$

then, according to a well known theorem of KAKEYA[2]), all the roots of (4) lie in the unit circle $|z| \leqq 1$. Indeed, multiplying (4) by $1-z$, we get the equation

$$a_0 + (a_1 - a_0) z + \ldots + (a_n - a_{n-1}) z^n - a_n z^{n+1} = 0 \tag{6}$$

where — in consequence of the inequalities (5) — the coefficient of z^{n+1} is negative, all the other coefficients are positive (or zero).

Now equation (6) has the only positive root $z = 1$, and, by theorem (A), all its other roots, i. e. all the roots of (4) lie in the circle $|z| \leqq 1$.

In the same way the following more exact form of theorem **2.** may be proved:

If the coefficients of (4) satisfy the inequalities

$$a_0 > 0,\ \varrho a_{\nu+1} - a_\nu > 0\ (\nu = 0, 1, \ldots, n-1),\ \varrho > 0, \tag{7}$$

then all the roots of (4) lie in the interior of the circle $|z| \leqq \varrho$.

If some of the differences $\varrho a_{\nu+1} - a_\nu$ vanish, some of the roots may fall on the boundary of the circle $|z| \leqq \varrho$.[3])

3. For the trinomic equation

$$f(z) = 1 - (n+1) z^n + n z^{n+1} = 0$$

$$F(z;\lambda) = |c_0| + \ldots + |c_{\nu-1}| z^{\nu-1} - |c_\nu| z^\nu + \lambda (|c_{\nu+1}| z^{\nu+1} + \ldots + |c_n| z^n),$$

the number of the roots of $F(z;\lambda) = 0$, whose moduli are smaller than $r_1(\lambda)$, is evidently the same for each positive value of $\lambda \leqq 1$. In the case of $\lambda = 0$ the equation

$$F(z;0) = |c_0| + \ldots + |c_{\nu-1}| z^{\nu-1} - |c_\nu| z^\nu = 0$$

has all its roots in the circle $|z| \leqq r_1(0)$, consequently $F(z;1) = 0$, i. e. the equation (1) has ν roots in the circle $|z| \leqq r_1$, q. e. d.

[2]) S. KAKEYA, On the limits of the roots of an algebraic equation with positive coefficients, *The Tôhoku Math. Journal*, **2** (1912), pp. 140–142.

[3]) For instance in the case of the equation

$$1 + z + 2z^2 + 2z^3 = 0$$

the differences $a_1 - a_0$, $a_3 - a_2$ vanish and one of the roots $z_1 = -1$, $z_2 = \frac{i}{\sqrt{2}}$, $z_3 = -\frac{i}{\sqrt{2}}$ falls on the boundary of the circle $|z| < 1$.

with the double root $z=1$ there is

$$f(z)=(1-z)^2(1+2z+\ldots+nz^{n-1}),$$

consequently all the roots lie in the circle $|z|\leqq 1$.

For the more general trinomic equation

$$f(z)=m-(n+m)z^n+nz^{n+m}=0$$

with the double root $z=1$ there is

$$f(z)=(1-z)^2\{m+2mz+\ldots+(n-1)mz^{n-2}+ \\ +nmz^{n-1}+n(m-1)z^n+\ldots+nz^{n+m-2}\}.$$

The coefficients in the second factor are all positive, and increasing up to the n-th term, and decreasing from there. For this equation, however, it is known,[4]) that $n-1$ of its roots lie inside, $m-1$ roots outside the unit circle. It seems natural to ask, whether for any equation, whose coefficients are positive and increasing up to a certain term, and decreasing from there, a similar statement holds.

The contrary is shown by the simple example

$$z^2+2\alpha z+\alpha^2=0,\quad 1/2<\alpha<2$$

whose double root $z=-\alpha$ may lie inside as well as outside the unit circle, the coefficients satisfying the inequalities

$$1<2\alpha>\alpha^2.$$

4. The sufficient condition for a similar separation of the roots, as it occurs in the case of the trinomic equation, is the following:

If the coefficients of the equation

$$a_0+a_1z+\ldots+a_nz^n=0 \tag{4}$$

are positive and if both sequences

$$\begin{matrix} 0,\ a_0,\ a_1\ldots a_m \\ a_m,\ a_{m+1},\ldots a_n,\ 0 \end{matrix} \tag{8}$$

are convex, i. e. if they satisfy the inequalities

$$a_{\nu-1}-2a_\nu+a_{\nu+1}\geqq 0 \text{ for } \nu=0,1,\ldots,m-1,m+1,\ldots,n, \tag{8'}$$
$$(a_{-1}=a_{n+1}=0)$$

---

[4]) G. Herglotz, Über die trinomische Gleichung, *Leipziger Berichte*, **74** (1922), pp. 1—8. See also E. Egerváry, A trinom egyenletről, *Math. és phys. lapok*, **37** (1930), pp. 30—57.

then m roots of (4) *are in the region* $|z| \leqq 1$, $n-m$ *in the region* $|z| \geqq 1$.

I shall prove the theorem in the following, more exact form:

If the coefficients of the equation (4) *satisfy the inequalities*

$$a_\nu > 0 \text{ for } \nu = 0, 1, 2, \ldots n, (a_{-1} = a_{n+1} = 0),$$
$$\mathrm{P}\varrho a_{\nu+1} - (\mathrm{P}+\varrho) a_\nu + a_{\nu-1} > 0 \text{ for } \nu = 0, 1, \ldots m-1, m+1, \ldots n, \quad (9)$$
$$\mathrm{P} \geqq \varrho > 0$$

then m roots are in the open region $|z| < \varrho$, $n-m$ *roots are in the open region* $|z| > \mathrm{P}$.

Indeed, multiplying (4) by $(\mathrm{P}-z)(\varrho - z)$, we get

$$\mathrm{P}\varrho a_0 + [\mathrm{P}\varrho a_1 - (\mathrm{P}+\varrho) a_0] z + [\mathrm{P}\varrho a_2 - (\mathrm{P}+\varrho) a_1 + a_0] z^2 + \ldots +$$
$$+ [\mathrm{P}\varrho a_{m+1} - (\mathrm{P}+\varrho) a_m + a_{m-1}] z^{m+1} + \ldots + \quad (10)$$
$$+ [-(\mathrm{P}+\varrho) a_n + a_{n-1}] z^{n+1} + a_n z^{n+2} = 0.$$

By the inequalities (9) all the coefficients are positive, except the coefficient of z^{m+1}, which must be evidently negative; consequently the equation (10) has, by the theorem of Pellet, $m+1$ roots in the interior or on the boundary of the region $|z| < \varrho$ and $n-m+1$ roots in the region $|z| > \mathrm{P}$ or on its boundary.

Hence, the original equation (4) has m roots in the interior of the region $|z| < \varrho$ and $n-m$ roots in the open region $|z| > \mathrm{P}$; the boundaries being exluded, because the coefficients a_ν satisfy for sufficiently small $\varepsilon > 0$ the inequalities (9) for $\mathrm{P}+\varepsilon$ and $\varrho - \varepsilon$ also.

If some of the second differences $\mathrm{P}\varrho a_{\nu+1} - (\mathrm{P}+\varrho) a_\nu + a_{\nu-1}$ vanish, some of the roots may fall on the boundaries of the regions $|z| < \varrho$, $|z| > \mathrm{P}$.[5])

5. It may be mentioned, that the last theorem contains the theorem of Kakeya, as a special case.

Indeed, if the coefficients of (4) satisfy the inequalities

$$a_0 > 0, \ \varrho a_{\nu+1} - a_\nu > 0 \text{ for } \nu = 1, 2, \ldots, n-1, \ \varrho > 0 \quad (7)$$

---

[5]) For instance in the case of the equation

$$3 + 6z + 4z^2 + 2z^3 + z^4 = 0$$

the second differences $a_1 - 2a_0 + 0$, $a_3 - 2a_2 + a_1$, $0 - 2a_4 + a_3$ vanish and two of the roots: $z_1 = z_2 = -1$, $z_3 = i\sqrt{3}$, $z_4 = -i\sqrt{3}$ fall on the boundary of the regions $|z| < 1$, $|z| > 1$.

then, for a sufficiently large positive value of P

$$\mathsf{P}\varrho a_0 > 0,$$
$$\mathsf{P}(\varrho a_1 - a_0) - \varrho a_0 = \mathsf{P}\varrho a_1 - (\mathsf{P} + \varrho) a_0 > 0,$$
$$\mathsf{P}(\varrho a_{\nu+1} - a_\nu) - (\varrho a_\nu - a_{\nu-1}) = \mathsf{P}\varrho a_{\nu+1} - (\mathsf{P}+\varrho) a_\nu + a_{\nu-1} > 0,$$
$$(\nu = 1, 2, \ldots n-1)$$

thus, the conditions (9) are satisfied for $m = n$, consequently the equation (4) has n roots in the interior of the circle $|z| < \varrho$.

If some of the differences $\varrho a_{\nu+1} - a_\nu$ vanish, then applying the former proof for $\varrho + \varepsilon\,(\varepsilon > 0)$, we get, that the equation (4) has all its roots in the interior or on the boundary of the circle $|z| < \varrho$.

(Received July 5, 1930.)