# An Elliptic System Corresponding to Poisson's Equation.

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## Introduction.

It is evident that the equations

$$\text{(1)} \qquad \frac{\partial \varphi}{\partial x} - \frac{\partial \theta}{\partial y} = u(x, y), \quad \frac{\partial \varphi}{\partial y} + \frac{\partial \theta}{\partial x} = v(x, y)$$

bear much the same relation to the CAUCHY—RIEMANN equations

$$\text{(2)} \qquad \frac{\partial \varphi}{\partial x} - \frac{\partial \theta}{\partial y} = 0, \quad \frac{\partial \varphi}{\partial y} + \frac{\partial \theta}{\partial x} = 0$$

that Poisson's equation

$$\text{(1')} \qquad \nabla^2 \psi = p(x, y)$$

bears to the equation of Laplace $\nabla^2 \psi = 0$. In fact, if $u$ and $v$ are sufficiently differentiable, $\varphi$ and $\theta$ satisfy (1'), with $p = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ and $p = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$, respectively. But this notion does not provide an adequate treatment of (1), precisely because it involves in an essential manner the derivatives of $u$ and $v$.

On the other hand, if we let $U(x, y)$ be a function such that $\nabla^2 U = u(x, y)$ and $V(x, y)$ such that $\nabla^2 V = v(x, y)$, the functions

$$\text{(3)} \qquad \varphi_1 = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}, \quad \theta_1 = \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}$$

are formal solutions of (1), as is seen by direct substitution.

In the following, we suppose that $u, v$ are bounded and measurable in the LEBESGUE sense, in a bounded region $T$; and we consider especially the potentials

$$(4)\quad \begin{aligned} U(x, y) = U(M) &= -\frac{1}{2\pi}\int_T \log\frac{1}{r}\, u(P)\, d\sigma_P, \\ V(M) &= -\frac{1}{2\pi}\int_T \log\frac{1}{r}\, v(P)\, d\sigma_P, \end{aligned}$$

with $r = MP$. We show, under certain general conditions, that if (1) has solutions at all, a particular pair of solutions is then furnished by (3), with (4), and the difference between these and any other pair of solutions is a pair of conjugate harmonic functions. Hence an inequality on the functions $\varphi_1$, $\theta_1$ has a valid application beyond the range of a hypothesis on the given functions $u, v$ such as would be sufficient to establish directly the existence of solutions.[1])

We note, in passing, the special case where $u$ and $v$ do satisfy conditions sufficient for the existence of first order derivatives of $\varphi_1$ and $\theta_1$.

*If $u, v$ are bounded in $T$, and satisfy at each point $M$ of $T$ a Hölder condition*

$$|u(P) - u(M)| \leq A(M)\,\overline{MP}^\alpha, \quad |v(P) - v(M)| \leq A(M)\,\overline{MP}^\alpha, \quad \alpha > 0,$$

*the functions $\varphi_1$, $\theta_1$ given by* (3), *with* (4), *are solutions of* (1).

It is well known that under these conditions $\frac{\partial^2 U}{\partial x^2}$ and $\frac{\partial^2 U}{\partial y^2}$ exist at $M$ and that $\nabla^2 U = u$. It may be verified easily that the same method of proof,[2]) and even more simply, shows that $\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial y}\right)$ and $\frac{\partial}{\partial y}\left(\frac{\partial U}{\partial x}\right)$ exist, and are equal. With this and the corresponding facts about $V$, the explicit differentiation becomes permissible, and the values given by (3) satisfy (1).

## 1. Properties of $\varphi_1$, $\theta_1$ derived from potentials.

Return now to the more general situation in which $u$ and $v$ are not continuous. Since $u(P) = u(x, y)$, $v(P) = v(x, y)$ are

[1]) This note is an exposition of the contents of the author's letter of September 1931 to Professor T. Radó, who inquired about the existence of such inequalities, and suggested that the results might be interesting to readers of these Acta.

[2]) O. D. Kellogg, *Foundations of Potential Theory* (Berlin, 1929), p. 153.

bounded, the functions $U, V$ defined by (4) and their first partial derivatives

$$\frac{\partial U}{\partial x} = -\frac{1}{2\pi}\int_T \frac{\cos(x,r)}{r} u(P)\,d\sigma, \qquad (5)$$
$$\frac{\partial U}{\partial y} = -\frac{1}{2\pi}\int_T \frac{\cos(y,r)}{r} u(P)\,d\sigma$$

exist and are continuous at every point of the finite plane. Hence the functions defined by (3), with (4), are continuous, and have the values

$$\varphi_1(M) = -\frac{1}{2\pi}\int_T \frac{1}{r}\left\{\cos(x,r)\,u(P) + \cos(y,r)\,v(P)\right\} d\sigma \qquad (6)$$
$$\theta_1(M) = -\frac{1}{2\pi}\int_T \frac{1}{r}\left\{\cos(x,r)\,v(P) - \cos(y,r)\,u(P)\right\} d\sigma.$$

But, if $|u(P)| < N$, $|v(P)| < N$ in $T$,

$$|\cos(x,r)\,u(P) + \cos(y,r)\,v(P)| < N\sqrt{2},$$

and $\int_T \frac{d\sigma}{r}$ is not greater than the same integral extended over a circle with center $M$ and the same measure as $T$; that is, $\leq 2\pi\sqrt{\frac{\text{meas. } T}{\pi}}$. Hence

$$(7) \qquad |\varphi_1(x,y)| < N\sqrt{\frac{2\,\text{meas. } T}{\pi}}, \quad |\theta_1(x,y)| < N\sqrt{\frac{2\,\text{meas. } T}{\pi}}.$$

A further relation is in evidence. The functions $\varphi_1$, $\theta_1$ satisfy the integral equations

$$\int_s \varphi\,dy + \theta\,dx = \int_\sigma u(P)\,d\sigma \qquad (8)$$
$$\int_s \theta\,dy - \varphi\,dx = \int_\sigma v(P)\,d\sigma,$$

for any simple rectifiable closed curve $s$ which, with its interior region $\sigma$, is interior to $T$ (or, when $u, v$ vanish identically outside $T$, for any such curves in the finite plane).

In fact, if $\sigma$ is made up of a finite number of non-overlapping rectangles, and $n$ denotes the interior normal, we have

$$\int_s \varphi_1 dy + \theta_1 dx = \int_s [-\varphi_1 \cos(x, n) + \theta_1 \cos(y, n)]\, ds$$

$$= -\int_s \left[\frac{\partial U}{\partial x} \cos(x, n) + \frac{\partial U}{\partial y} \cos(y, n)\right] ds +$$

$$+ \int_s \left[\frac{\partial V}{\partial x} \cos(y, n) - \frac{\partial V}{\partial y} \cos(x, n)\right] ds =$$

$$= -\int_s \frac{dU}{dn} ds + \int_s \frac{dV}{ds} ds,$$

the last integral vanishing, since $V, \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}$ are continuous. But

$$\int_{y_1}^{y_2} \frac{\partial U}{\partial x} dy = -\frac{1}{2\pi} \int_T d\sigma\, u(P) \int_{y_1}^{y_2} \frac{\cos(x, r)}{r} dy,$$

since the quantity

$$\int_{y_1}^{y_2} dy \int_T \frac{|\cos(x, r)|}{r} |u(P)|\, d\sigma$$

exists, and since for a fixed $x$, the function $\frac{\cos(x, r)}{r} u(P)$ is measurable in the three dimensional region $\{(\xi, \eta) \text{ in } T,\ y_1 < y < y_2\}$, where $P = (\xi, \eta)$.[8]) Hence, for any rectangle $s$

$$\int_s \frac{dU}{dn} ds = -\frac{1}{2\pi} \int_T d\sigma\, u(P) \int_s -\frac{\cos(x, r)}{r} dy + \frac{\cos(y, r)}{r} dx$$

$$= -\frac{1}{2\pi} \int_T d\sigma\, u(P) \int_s \frac{d}{dn} \log \frac{1}{r} ds = -\int_\sigma u(P)\, d\sigma.$$

Hence for any finite collection of non-overlapping rectangles, the first of equations (8) is established. The simple rectifiable curve is a limiting case, for the left hand member, by definition of the curvilinear integral, and for the right hand member, by the absolute continuity of $\int_\sigma u(P)\, d\sigma$. The second of equations (8) may be established in a similar manner.

[8]) Ch. de la Vallée Poussin, *Cours d'analyse infinitésimale*, vol. 2 (Paris, 1912), p. 122.

These results may be summarized as follows.

Theorem. *If $u, v$ are bounded in $T$ and summable in the Lebesgue sense over $T$, the functions $\varphi_1$. $\theta_1$, defined by* (3), *with* (4), *have the form* (6), *are bounded and continuous in the finite plane, and satisfy the inequalities* (7); *and within $T$ satisfy the integral equations* (8) *for curves $s$ which are contained with their interiors in $T$.*

## 2. The homogeneous integral equations.

The differences of two pairs of solutions of (8) are solutions of the homogeneous equations

$$\int_s \varphi\, dy + \theta\, dx = 0, \quad \int_s \theta\, dy - \varphi\, dx = 0. \tag{9}$$

Theorem. *A pair of continuous solutions of* (9), *for curves $s$ of the kind described, form a pair of conjugate harmonic functions in $T$.*

Consider any simply connected portion $T'$ of $T$. Let $Q$ be a fixed and $M$ a variable point in $T'$. Then, by the first of (9),

$$\int_Q^M \varphi\, dy + \theta\, dx = \Psi(M)$$

defines a single valued function of $M$ in $T'$, and since $\varphi$ and $\theta$ are continuous in $T'$,

$$\frac{\partial \Psi}{\partial x} = \theta, \quad \frac{\partial \Psi}{\partial y} = \varphi$$

$$\frac{d\Psi}{d\alpha} = \theta \cos(x, \alpha) + \varphi \cos(y, \alpha)$$

where $\frac{d\Psi}{d\alpha}$ is the directional derivative in the direction $\alpha$.

The second of equations (9) now becomes

$$\int_s \frac{\partial \Psi}{\partial x} dy - \frac{\partial \Psi}{\partial y} dx = 0,$$

so that

$$\int \frac{d\Psi}{dn} ds = 0.$$

But it is a well known theorem of BÔCHER that if $\Psi, \frac{\partial \Psi}{\partial x}, \frac{\partial \Psi}{\partial y}$ are continuous in $T'$ and $\int_s \frac{d\Psi}{dn} ds = 0$, for all circles $s$ in $T'$, then the derivatives of all orders of $\Psi$ exist and are continuous in $T'$ and $\Psi$ satisfies Laplace's equation.[4]) Consequently

$$\frac{\partial \varphi}{\partial x}=\frac{\partial^2 \Psi}{\partial x \partial y}=\frac{\partial \theta}{\partial y}, \quad \frac{\partial \varphi}{\partial y}=\frac{\partial^2 \Psi}{\partial y^2}=-\frac{\partial^2 \Psi}{\partial x^2}=-\frac{\partial \theta}{\partial x}$$

and $\varphi$ and $\theta$ are conjugate harmonic functions in $T'$. And any point of $T$ is interior to some $T'$. In this demonstration, rectangles or other special classes of curves may be used in place of circles.

## 3. Relations between differential and integral equations.

In general, in order to make connection between (1) and (8), mere continuity of $\varphi$ and $\theta$ is not enough; it is necessary to make some hypothesis of absolute continuity of $\varphi$ and $\theta$ with respect to the variables $x$ and $y$ separately, and it is desirable to make the assumption as a two dimensional condition in dealing with two-dimensional problems. We retain the requirement of continuity, as appropriate to the present problem, and add the following:

(A). Consider simple closed rectifiable curves, which with their interior regions lie in $T$ (in fact, rectangles suffice) and write $F(\sigma)=\int_s \varphi\, dy$, $G(\sigma)=-\int_s \varphi\, dx$. The condition is that $F(\sigma)$, *$G(\sigma)$ define absolutely continuous functions of point sets in $T$.*

As a result of this condition, $\frac{\partial \varphi}{\partial x}$ and $\frac{\partial \varphi}{\partial y}$ exist almost everywhere and are summable in $T$, having almost everywhere the values of the point set derivatives of these respective functions of point sets; and for almost all $y$ the $\varphi$ is absolutely continuous in $x$, and for almost all $x$ it is absolutely continuous in $y$.[5])

[4]) M. Bôcher, On Harmonic Functions in Two Dimensions, *Proceedings of the American Academy of Sciences*, 41 (1905—6), pp. 577—583.

[5]) This condition was introduced and similar properties obtained in G. Evans, Fundamental Points of Potential Theory, *Rice Institute Pamphlet*, 7 (1920), pp. 252—329 (see p. 274), with $\varphi$ summable instead of continuous. In the restriced case in which $\varphi$ is continuous it is identical with the concept of „funzione di due variabili assolutamente continua" described by L. Tonelli, Sulla quadratura delle superficie, *Rendiconti della R. Accademia dei Lincei*, (6) 3 (1926, 1° sem.), pp. 633—638, and Sulle funzioni di due variabili assolutamente continue, *Memorie della R. Accademia delle Scienze dell'Istituto di Bologna* (Scienze fisiche), (8) 6 (1928—29), pp. 81—88. See also G. Evans, *The Logarithmic Potential* (New York, 1927), p. 146.

Theorem. *Let* $\varphi$ *and* $\theta$ *be continuous in* $T$ *and satisfy* (A), *and let* $s$ *be a curve of the kind already described. Then if* $\varphi, \theta$ *satisfy* (1) *almost everywhere they satisfy* (8) *for all* $s$, *and if they satisfy* (8), *in fact, merely for rectangles, they satisfy* (1) *almost everywhere.*

For, the equations (1), holding almost everywhere, imply

$$(10)\quad \int_{\sigma}\left(\frac{\partial\varphi}{\partial x}-\frac{\partial\theta}{\partial y}\right)d\sigma=\int_{\sigma}u\,(x,y)\,d\sigma,\ \int_{\sigma}\left(\frac{\partial\varphi}{\partial y}+\frac{\partial\theta}{\partial x}\right)d\sigma=\int_{\sigma}v\,(x,y)\,d\sigma,$$

since the first partial derivatives of $\varphi$ and $\theta$ are summable. And (10) implies (8), by (A), for any curve which, with its interior region, is contained in $T$. Conversely, (8) implies (10) by (A), and (10) implies (1) almost everywhere.

Corollary. *If the system* (1) *has a pair of solutions which are continuous and satisfy condition* (A), *then it has as solutions the particular functions* $\varphi_1, \theta_1$, *which satisfy the inequality* (7), *and also these functions plus an arbitrary pair of conjugate harmonic functions in* $T$.

For, by the theorem just given, such a pair must constitute solutions of (8). But if one pair of solutions of (8) satisfy (A), any pair must do so, since the pairs of differences are harmonic in $T$.

In conclusion, we remark that instead of (8) which are not much more general than the differential equations (1), we might consider equations of a different order of generality, namely

$$(11)\qquad \int_{s}\varphi\,dy+\theta\,dx=F(s),\ \int_{s}\theta\,dy-\varphi\,dx=G(s),$$

in which $\varphi$ and $\theta$ are merely summable, and $F(s)$, $G(s)$ are additive functions of curves, which correspond, on curves of continuity, to completely additive functions of point sets; the equations (11) are assumed to hold on „almost all" rectangles, or „almost all" curves of a certain class. By taking alternalely $F(s)\equiv 0$ and $G(s)\equiv 0$ the problem may be reduced to one in potential theory considered by the author.[6]

[6] Discontinuous Boundary Value Problems of the First Kind for Poisson's Equation, *American Journal of Mathematics*, 51 (1929), pp. 1—18.

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