On Continuability of Power Series.

By A. ZYGMUND in Wilno.

§ 1.

Let $\Sigma a_n z^n$ be an arbitrary power series whose radius of convergence is equal to 1. As regards the continuability of the series

$$\sum_{n=0}^{\infty} \varepsilon_n a_n z^n, \tag{1}$$

where $\{\varepsilon_n\}$ is a sequence of unit factors: $|\varepsilon_n| = 1$, the following theorems have been proved.

a) There exists a sequence of $\varepsilon_n = \pm 1$, for which (1) is not continuable across $|z| = 1$.[1])

b) For „almost all" sequences $\varepsilon_n = \pm 1$, the series (1) is not continuable across $|z| = 1$.[2])

c) If $\sqrt[n]{|a_n|} \to 1$, the set of sequences $\{\varepsilon_n\}$, $\varepsilon_n = \pm 1$ such that (1) can be continued outside $|z| = 1$ is at most enumerable.[3])

d) If ε_n are arbitrary (complex) unit factors: $\varepsilon_n = e^{2\pi i x_n}$ $(0 \leq x_n < 1)$ then for „almost all" $\{\varepsilon_n\}$ the functions (1) are not continuable.[4])

---

[1]) Theorem of PÓLYA. See e. g. E. LANDAU, *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie*, 2nd ed. (1929), p. 86—87.

[2]) See R. E. A. C. PALEY and A. ZYGMUND, On some series of functions (3), *Proc. Cambridge Phil. Soc.*, **28** (1932), p. 190—205, esp. p. 201, Theorem XI.

[3]) F. HAUSDORFF, Zur Verteilung der fortsetzbaren Potenzreihen, *Math. Zeitschrift*, 4 (1919), p. 98—103.

[4]) H. STEINHAUS, Über die Wahrscheinlichkeit dafür, daß der Konvergenzkreis einer Potenzreihe ihre natürliche Grenze ist, *Math. Zeitschrift*, 31 (1929), p. 408—416.

„Almost all" in *b)* is meant as follows. We consider the sequence of (RADEMACHER's) functions $\varphi_0(t), \varphi_1(t) \ldots, \varphi_n(t), \ldots$, where $\varphi_n(t) = \text{sign} \sin (2^{n+1}\pi t)$ and put

$$f_t(z) = \sum_{n=0}^{\infty} a_n \varphi_n(t) z^n. \tag{2}$$

Proposition *b)* asserts that for almost every t the functions $f_t(z)$ have the circle $|z| = 1$ as their natural boundary. As regards „almost all" in *d)* we refer the reader to the paper of STEINHAUS.

It is clear that *a)* is included in *b)* (but not in *d)*). The condition $\sqrt[n]{|a_n|} \to 1$ is obviously essential for the validity of *c)*. Propositions *b)* and *d)* cannot be compared. Theorem *c)* is false if ε_n, $|\varepsilon_n| = 1$, is permitted to assume complex values.

The proof of *b)* given in the paper referred to can be easily extended to some more general classes of $\{\varepsilon_n\}$ and, in particular, it gives *d)*. However the proof of the latter theorem given by STEINHAUS and using some specific properties of the space of sequences $\{x_n\}$ $(0 \leqq x_n < 1)$ is more elementary. It is therefore natural to apply that method to the proof of theorem *b)* and this is just the purpose of the present paper. As regards the following proof, two remarks may be added.

1°. Although the proof is more elementary than the original proof in the paper quoted under [2]), the method seems to be less powerful. In particular it does not give the corresponding theorem for DIRICHLET series.[5]) 2°. The reader acquainted with STEINHAUS' paper will notice, that, although we utilise some of its ideas, the changes that had to be made are not completely trivial (see also § 4 below).

§ 2.

We begin with a few general remarks on the structure of linear sets of points. Let $t = 0, \alpha_1 \alpha_2 \ldots \alpha_n \ldots$ be the dyadic development (the only one if t is not dyadically rational) of an arbitrary number in the interval $0 \leqq t \leqq 1$. Let Q_n $(n = 1, 2, \ldots)$ be one of the two operations which may be performed upon α_n: leaving α_n unaltered or replacing α_n by $1 - \alpha_n$. Let for an arbitrary sequence $Q_1, Q_2, \ldots, Q_n \ldots$ the operation $Q_1 Q_2 \ldots Q_n \ldots$ be denoted by Q. Q transforms every number $t = 0, \alpha_1 \alpha_2 \ldots$ into another

[5]) See PALEY and ZYGMUND, loc. cit., Theorem XXVII.

number $Q(t) = 0, \beta_1\beta_2 \ldots$ If we reject from the interval $I = (0, 1)$ the (enumerable) set D of numbers t such that either t or $Q(t)$ is dyadically rational, the rest $I - D$ is transformed by Q into itself. Given an arbitrary set E of points t we shall denote by $Q(E)$ (image of E) the set of points $Q(t)$ when $t \in E - D$.

Lemma. For an arbitrary measurable set E of points t, the set $Q(E)$ is measurable and $|E| = |Q(E)|$.[6])

The lemma is obviously true if E is an interval of the form $(k2^{-N}, (k+1)2^{-N})$ $(N = 1, 2, \ldots;\ k = 0, 1, \ldots, 2^N - 1)$. Every open set can be represented as the sum of a finite or enumerable system of such intervals. Hence the truth of the lemma follows for open, and, consequently, passing to the complements, also for closed sets. It is sufficient to notice that any measurable set is contained between two sets, one open, the other closed, with measures differing as little as we please.

Let $E(\alpha, \beta)$ denote the set of t for which the function $f_t(z)$ $(z = re^{i\theta})$ posesses at least one singular point on the arc $\alpha \leqq \theta \leqq \beta$ $(r = 1)$. It is not difficult to see that $E(\alpha, \beta)$ is always measurable. Moreover, (the proposition and the proof are well known) $E(\alpha, \beta)$ is either of measure 0 or 1. In fact, rejecting a finite number of terms in the series (2) we do not change $E(\alpha, \beta)$. Consequently if γ is an arbitrary dyadically rational number and $E_\gamma(\alpha, \beta)$ the set $E(\alpha, \beta)$ translated by γ (and taken mod. 1), we have $E_\gamma(\alpha, \beta) = E(\alpha, \beta)$. If $E(\alpha, \beta)$ and its complement were both of positive measure we could, choosing γ suitably, bring a point of density of $E(\alpha, \beta)$ as near as we please to a point of density 0, what is, of course, impossible.

§ 3.

Now, passing to our problem, we prove that $E(\alpha, \beta)$ has the same measure as $E(\alpha + \pi, \beta + \pi)$. In fact, $f_t(z)$ possesses a singular point on the arc $(\alpha + \pi, \beta + \pi)$ if and only if the function $f_t(-z) = \Sigma a_n z^n (-1)^n \varphi_n(t)$ possesses one on (α, β). But $(-1)^n \varphi_n(t) = \varphi_n(t')$, $f_t(-z) = f_{t'}(z)$, where t' can be obtained from t by a transformation Q, and the assertion follows from Lemma. In particular, as every $f_t(z)$ has at least one singular point on $|z| = 1$, at least one of the sets $E(\alpha, \alpha + \pi)$, $E(\alpha + \pi, \alpha + 2\pi)$ is of positive measure. It follows that both sets are of measure 1.

[6]) By $|E|$ we denote the (Lebesgue) measure of E.

Let $N>0$ be an arbitrary integer and (α, β) an arc of length $\leq 2\pi/2^{-N}$, I_k^N $(k=0, 1, \ldots, 2^N-1)$ the arc (α, β) translated by $2\pi k/2^N$. We are going to prove that if almost all $f_t(z)$ have singular points on one of the arcs $I_0^N, I_1^N, I_2^N, \ldots$ the same may be said of the other arcs. We have already proved this for $N=1$; let us suppose the assertion true for $N-1$. Since $\beta-\alpha\leq 2\pi/2^N<2\pi/2^{N-1}$, almost all functions $f_t(z)$ have singular points either on the arcs $I_0^N, I_2^N, \ldots, I_{2^N-2}^N$ or on $I_1^N, I_3^N, \ldots, I_{2^N-1}^N$, say, on the former. Suppose, contrarily to what we are trying to prove, that almost all $f_t(z)$ are regular on the arcs $I_1^N, I_3^N, \ldots$, i. e. that for almost all t, the functions $f_t(\omega z), f_t(\omega^3 z), f_t(\omega^5 z), \ldots$ where $\omega=\exp(2\pi i/2^N)$ are regular on I_0^N. Let $n_\lambda^{(r)}=2^{N-1}\lambda+r$ $(0\leq r<2^{N-1}, \lambda=0, 1, 2, \ldots)$. Then, for ν odd,

$$(3)\qquad f_t(\omega^\nu z)=\sum_n \omega^{\nu n} a_n z^n \varphi_n(t)=$$
$$=S_0+\omega^\nu S_1+\omega^{2\nu} S_2+\ldots+\omega^{\nu(2^{N-1}-1)} S_{2^{N-1}-1}$$
$$(\nu=1, 3, 5, \ldots, 2^N-1)$$

where

$$S_r=\sum_\lambda a_{n_\lambda^{(r)}} z^{n_\lambda^{(r)}} (-1)^\lambda \varphi_{n_\lambda^{(r)}}(t).$$

Since the determinant of the linear system of equations (3) is not zero, the functions $S_0, S_1, \ldots, S_{2^{N-1}-1}$ are linear combinations of $f_t(\omega z), f_t(\omega^3 z)\ldots$ and, consequently, they are, as well as their sum $S=S_0+S_1+\ldots+S_{2^{N-1}-1}$, regular on I_0^N for almost every t. But $S=S_t(z)=f_{t'}(z)$, where $t'=Q(t)$, hence, contrarily to our hypothesis, almost all $f_t(z)$ are regular on I_0^N.

As at least one of the sets $E(2k\pi/2^N, 2(k+1)\pi/2^N)$ $(k=0, 1, \ldots, 2^N-1)$ is of positive measure, i. e. of measure 1, almost all $f_t(z)$ have singular points on every arc I_k^N. It follows that almost all $f_t(z)$ have $|z|=1$ as their natural boundary.

§ 4.

It is natural to inquire whether the method of the proof can be applied to analogous problems. Let $\mathfrak{K}$ be a class of power series $\mathfrak{P}$, and suppose that, if $\mathfrak{P}$ belongs to $\mathfrak{K}$, every series that may be obtained from $\mathfrak{P}$ by multiplying the coefficients by *complex* unit factors ε_n or by changing a finite number of the coefficients of $\mathfrak{P}$, also belongs to $\mathfrak{K}$. Let P be a property such, that, for any function f of $\mathfrak{K}$, there exists a direction $\arg z = \theta_0$, for which $f(z)$ possesses property P in every angle $\theta_0 - \varepsilon \leq \arg z \leq \theta_0 + \varepsilon$, and that changing of a finite number of terms in the development of f, θ_0 remains unaltered. Then an argument completely analogous to that used by STEINHAUS in his paper referred to, shows that (in STEINHAUS' sens) „almost all" functions obtained from $f(z)$ by introducing complex unit factors ε_n possess property P in every angle $\alpha \leq \arg z \leq \beta$. For instance, if we take for $\mathfrak{K}$ the class of all power series with radius of convergence equal to 1, and for P existence of singularities on a given arc of the circle $|z| = 1$, we get proposition *d)*. If $\mathfrak{K}$ is the class of integral functions and P existence of directions J (JULIA) we get the theorem (which, it seems, has never been explicitely stated), that for „almost all" series (1) with complex ε_n, $|\varepsilon_n| = 1$, every direction is J.[7])

As regards the factors $\varepsilon_n = \pm 1$, the problem, propounded by PÓLYA[8]) in a little weaker form, requiring only existence of *one* sequence $\{\varepsilon_n\}$, looks more difficult. In the above proof (§ 3) we utilised the fact that, if two functions are regular on an arc, so is their sum. The corresponding theorem for directions not J is not true and the problem remains unsolved. Only in the case of functions of infinite order it is not difficult to prove that for almost every sequence of $\varepsilon_n = \pm 1$ PÓLYA's hypothesis is true.

(Received February 11, 1933.)

---

[7]) A direction $\arg z = \theta_0$ is J if in every angle $(\theta_0 - \varepsilon, \theta_0 + \varepsilon)$ $f(z)$ assumes every value, with one possible exception, infinitely many times. It has been proved by VALIRON (see M. BIERNACKI, Sur la théorie des fonctions entières, *Bull. de l'Acad. Polonaise,* (A) 1929, p. 529–570, esp. p. 546 sq. where a different proof is given) that for every integral function $f(z)$ there exists a direction θ_1 (direction J^*) such that, whatever rational function $R(z)$, with one possible exception, $f(z) - R(z)$ vanishes in every angle $(\theta_1 - \varepsilon, \theta_1 + \varepsilon)$ infinitely many times. It is obvious that changing a finite number of coefficients of $f(z)$ does not move J^* directions. Hence in the theorem stated above we may replace J by J^*. It must be added that the proofs of measurability of sets analogous to $E(\alpha, \beta)$ (considered above) are, for directions J and J^*, a little troublesome.

[8]) G. PÓLYA, Untersuchungen über Lücken und Singularitäten von Potenzreihen, *Math. Zeitschrift,* **29** (1929), p. 549—640.