Functions of Self-Adjoint Transformations in Hilbert Space.

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Introduction.

The theory of functions of general linear and more specially of self-adjoint operators in HILBERT space is by no means new. More than twenty years ago V. VOLTERRA developed the notion of an analytic function of an operator[1]). Subsequently, in order to establish the spectral resolution of a bounded self-adjoint operator, F. RIESZ introduced the theory of continuous and semi-continuous functions of such operators[1]). More recently J. NEUMANN and M. STONE treated the theory of a general function of a self-adjoint operator[2]). The latter authors operate with bilinear forms rather than the operators themselves; this means that numerical LEBESGUE—STIELTJES integration may be introduced but necessitates a subsequent reinterpretation of the results obtained in terms of transformations[3]).

It is known[4]) that the introduction of bilinear forms is not necessary but that the notions involved can be developed in the direction suggested by one's intuition, that is by dealing with the

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1) See F. RIESZ, *Les systèmes d'équations linéaires à une infinité d'inconnues* (Paris, 1913), p. 130, text and foot-note.

2) J. v. NEUMANN, Über Funktionen von Funktionaloperatoren, *Annals of Math.*, 32 (1931), pp. 191—226; M. H. STONE, *Linear Transformations in Hilbert Space* (New York, 1932), Chapt. VI.

3) We use the words transformation and operator interchangeably.

4) See *these Acta*, 6 (1934), p. 204. I take this occasion to state that the many suggestions of Prof. RIESZ have been extremely valuable during the preparation of this paper.

operators directly. The purpose of this paper is not so much to give the details of this development as to suggest that the intuitive approach is also the shortest. We establish a theory of measure and integration for operators which reflects faithfully the theory of LEBESGUE integration. Some interesting differences necessarily arise; we note for instance that in our theory the measure of a set virtually determines the set and that the measure of the sum of any two measurable sets is equal to the sum of their measures.

§ 1. Let A be a linear transformation defined over a linear manifold D_A dense in HILBERT space $\mathfrak{H}$. Consider the set of all pairs of elements $[g, g^*]$ such that $(Af, g) = (f, g^*)$ for all f in D_A. Here the symbol (φ, ψ) represents as usual the inner product of the elements φ and ψ. We construct a transformation A^* defined over the set D_{A^*} of all elements g above by means of the equation $A^* g = g^*$. A^* is known as the operator adjoint to A. If $D_A = D_{A^*}$ and if $Af = A^* f$ throughout D_A we say that A is a self-adjoint operator. It is well known that for any self-adjoint operator A there exists a family of projections[5]) $E(\lambda)$, $-\infty < \lambda < \infty$, called the resolution of the identity of A, having properties a) $E(\lambda)E(\mu) = E(\mu)E(\lambda) = E(\lambda)$ for $\lambda \leqq \mu$; b) $E(\lambda)f \to 0$ or f as $\lambda \to -\infty$ or ∞; and c) for $\mu > \lambda$, $E(\mu)f \to E(\lambda)f$ as $\mu \to \lambda$. The transformation A is defined for the element f if and only if the Stieltjes integral $\int_{-\infty}^{\infty} \lambda^2 d\|E(\lambda)f\|^2$ converges and then

$$\|Af\|^2 = \int_{-\infty}^{\infty} \lambda^2 d\|E(\lambda)f\|^2. \tag{1}$$

Furthermore

$$Af = \int_{-\infty}^{\infty} \lambda dE(\lambda)f. \tag{2}$$

The integral (2) is analoguous to a Stieltjes integral; its meaning may be clarified as follows: We subdivide the real axis into denumerably many intervals closed on the right Δ_i by means of the points $\ldots a_{-n} < \ldots < a_0 \ldots < a_n \ldots$ $(a_{-n} \to -\infty,\ a_n \to \infty)$; here Δ_i is the set $a_{i-1} < x \leqq a_i$. Let $E(\Delta_i) = E(a_i) - E(a_{i-1})$; let c_i be any point in Δ_i and f any element in $\mathfrak{H}$ for which (1) con-

[5]) A projection is a self-adjoint transformation E defined throughout $\mathfrak{H}$ such that $E^2 = E$.

verges. Form the sum $\sum_{i=-\infty}^{\infty} c_i E(\Delta_i) f$; as the maximum length of the Δ_i approaches zero the sum converges strongly to an element in $\mathfrak{H}$ which we denote by Af[6]).

We add a few remarks on self-adjoint transformations. We have stated that any operator of this type possesses a resolution of the identity. Conversely, any resolution (of the identity) gives rise to a self-adjoint operator. Furthermore, it may be verified in (1) that a self-adjoint operator is defined for all f in $\mathfrak{H}$ if and only if there exists a constant $C > 0$ such that $E(C) = 1$ (the identity operator) and $E(-C) = 0$ (the zero operator). We then have $\|Af\| \leq C \|f\|$. Such transformations are called bounded. If no such constant exists, we shall often speak of the transformation as unbounded. Finally a self-adjoint transformation possesses the property of closure, that is, if f_n is a sequence of elements in $\mathfrak{H}$ for which Af_n exists and if $f_n \to f$, $Af_n \to g$ then Af is defined and $Af = g$. From this point forward, the unmodified term transformation will indicate self-adjoint transformation unless the contrary is indicated.

§ 2. In order to discuss the notion of a function of an operator A, we shall first study the relation with respect to the resolution $E(\lambda)$ of A of certain linear sets of points to closed linear manifolds in $\mathfrak{H}$. We introduce notation suitable to our purpose: If M_α denotes any closed linear manifold of a set of such manifolds, the expression $\sum_\alpha M_\alpha$ will indicate the smallest closed linear manifold containing all M_α. The expression $\prod_\alpha M_\alpha$ will as usual denote the intersection of all the M_α, that is, the largest closed linear manifold contained in all M_α. To every M_α we may associate a projection P_α whose range is M_α; that is, P_α transforms an arbitrary f in $\mathfrak{H}$ into its projection on M_α. If the P_α are commutative in pairs we shall say that the M_α are commutative. If a manifold M_1 contains a manifold M_2, $M_1 - M_2$ will denote the manifold of those elements in M_1 orthogonal to M_2. We now introduce

Lemma 1: *Let M_n be a monotone decreasing sequence of*

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[6]) We are repeating the argument presented by F. Riesz in a paper: Über die linearen Transformationen des komplexen Hilberschen Raumes, *these Acta*, 5 (1932), pp. 23—54, especially pp. 48—51.

manifolds, $M_n \supseteqq M_{n+1}$ $(n=1, 2, \ldots)$ *and* P_n *be the corresponding projections. Let* $M = \prod_{\alpha=1}^{\infty} M_\alpha$ *and let* P *correspond to* M. *Then* $\|(P_n - P)f\| \to 0$ *for all* f *in* $\mathfrak{H}$. *Furthermore,* P *is commutative with* P_n *and with any bounded operator* S *commutative with* P_n.

It is elementary that the operators P_n converge to a limit which is a projection; let us denote it by P'. Then $M' \supseteqq M$ since $M_n \supseteqq M$ for all n. Let f belong to $M' - M$, the manifold whose definition has been given above. If $\|f\| = c > 0$, there exists an m such that f is not in M_n, $n \geqq m$. Let $\|P_n f\| = a$ where $a < c$. Then $\|P'f\| \leqq a < c$. This means that $c = 0$ and $M = M'$.

The statement as to permutability is immediate. M is contained in all M_n hence permutable with them. Furthermore from $SP_n = P_n S$ follows $SP = PS$ since S is bounded and hence continuous. We now state

Lemma 2: *Let* $\{M_\alpha\}$ *denote any set of commutative manifolds which contains the product of any two of its members. If* $M = \prod_\alpha M_\alpha$ *there exists a denumerable subset* $\{M_n\}$ *of* $\{M_\alpha\}$ *such that* $M = \prod_{n=1}^{\infty} M_n$. *We may choose the* M_n *so that* $M_n \supseteqq M_{n+1}$.

We may assume without restriction of generality that $M = 0$[7]). Let f be in $\mathfrak{H}$ and let us suppose that the lower bound of $\|P_\alpha f\| = c > 0$. We choose a sequence M_n such that $P_n f \to g$, $\|g\| = c$; we may and shall assume that $M_n \supseteqq M_{n+1}$. Now some M_α does not contain g. We use an argument very similar to that used in lemma 1 and see that the lower bound of $\|P_\alpha f\|$ must be zero.

Let now $f_1, f_2, \ldots$ denote a sequence of elements ewerywhere dense in $\mathfrak{H}$. We choose manifolds M_{ij} such that $\lim_{i=\infty} P_{ij} f_j = 0$ $(j = 1, 2, \ldots)$. Arranging the M_{ij} in some linear order, we let M_n correspond to the product of the first n of the M_{ij} and have the result stated in the lemma. We note that the result is not valid in non-separable spaces.

For the remainder of § 2, we fix our attention on a given resolution $E(\lambda)$. Let G be an open set on the real axis. Let γ be a half-closed interval $a < x \leqq b$ contained in G. We define $E(\gamma) = E(b) - E(a)$; we let $M(\gamma)$ denote the range of $E(\gamma)$. Finally we define

[7]) If $M \neq 0$, we replace M_α by $M_\alpha - M$; then $\prod_\alpha (M_\alpha - M) = 0$.

$M(G)=\sum_{\alpha} M(\gamma_\alpha)$ where the sum is to be carried out over all $\gamma_\alpha \subseteqq G$. We note that if $G_1 \supseteqq G_2$, $M(G_1) \supseteqq M(G_2)$. If G denotes an open interval, the projection $E(G)$ associated with $M(G)$ is commutative with any bounded operator commutative with $E(\lambda)$. For $E(G)$ is the limit of a monotone increasing sequence of projections of the type $E(\gamma)$. Similarly, if G is any open set, $E(G)$ is commutative with any bounded operator commutative with $E(\lambda)$.

Let $G=\Sigma_{\alpha} G_\alpha$ where all the sets are open; then $M(G)=\sum_{\alpha} M(G_\alpha)$. For since $G \supseteqq G_\alpha$, $M(G) \supseteqq \sum_{\alpha} M(G_\alpha)$. On the other hand every half-closed interval γ in G can be covered by a finite or denumerable number of such intervals belonging to the G_α, hence $M(G) \subseteqq \sum_{\alpha} M(G_\alpha)$. If $G=G_1 \bullet G_2$, then $M(G)=M(G_1) \bullet M(G_2)$. Clearly we have $M(G) \subseteqq M(G_1) \bullet M(G_2)$. In case G_1 and G_2 can each be expressed as the sum of a finite number of open intervals, the inequality may be erased. As every element in $M(G_1) \bullet M(G_2)$ may be approximated by elements in manifolds corresponding to open sets consisting of only a finite number of open intervals, each in G, we have $M(G) \supseteqq M(G_1) \bullet M(G_2)$.

Let G be an open set, $\overline{G}$ its complement on the real axis. Let $\{G_\alpha\}$ denote the set of open sets which contains $\overline{G}$. We note that $\{G_\alpha\}$ contains sets of the following type: We take the set G and suppress all but a finite number of open intervals; in the remaining set we replace each open interval by a closed one entirely interior to it. The complement of such a closed set then contains $\overline{G}$. This argument indicates that the manifolds $\Pi_{\alpha} M(G_\alpha)$ and $M(G)$ are orthogonal[8]). We introduce

Definition 1: *Let H be an arbitrary set, and let $\{G_\alpha\}$ be the set of all open sets containing H. Then the manifold $\Pi_{\alpha} M(G_\alpha)$ is called the exterior manifold-measure of H. If the exterior manifold-measure of H is orthogonal to the exterior manifold-measure of the complement of H we shall say that H is measurable and that its manifold-measure is $M(H)=\Pi_{\alpha} M(G_\alpha)$.*

[8]) Since for every f in $\mathfrak{H}$ and any open set G we may construct a sequence of closed sets $F_n \subset G$, each closed set consisting of a finite number of closed intervals, such that the projection of f on $M(\overline{F}_n)$ converges to the projection of f on $\mathfrak{H}-M(G)$. Here the set $\overline{F}_n$ denotes the complement of F_n.

We note that all open sets are measurable (with respect to $E(\lambda)$ of course). This fact gives significance to

Theorem 1: *Let H_n be any sequence of measurable sets, $M(H_n)$ their manifold-measures. Then $H = \prod_{\alpha=1}^{\infty} H_\alpha$ is measurable and $M(H) = \prod_{\alpha=1}^{\infty} M(H_\alpha)$. Similarly, $H' = \sum_{\alpha=1}^{\infty} H_\alpha$ is measurable and $M(H') = \sum_{\alpha=1}^{\infty} M(H_\alpha)$.*

Applying lemma 2, we pick open sets $G_{m,n}$ such that $G_{m,n} \supseteqq H_n$ $(m = 1, 2, \ldots)$ and $M(H_n) = \prod_{\alpha=1}^{\infty} M(G_{\alpha,n})$. We note that the product of any finite number of the $G_{m,n}$ yields an open set which contains H. Using the earlier established fact that the measure of a product of a finite number of open sets is equal to the product of their measures, we see that the exterior measure $M_{ex}(H)$ of H satisfies the relation

$$M_{ex}(H) \subseteqq \prod_{\alpha,\beta=1}^{\infty} M(G_{\alpha,\beta}) = \prod_{\alpha=1}^{\infty} M(H_\alpha).$$

For the complement $\bar{H}$ of H, we know that $\bar{H} = \sum_{\alpha=1}^{\infty} (\bar{H}_\alpha)$. Since $\bar{H}_n$ as well as H_n is measurable, we may choose open sets $G'_{m,n}$ such that $G'_{m,n} \supseteqq \bar{H}_n$ $(m = 1, 2, \ldots)$ and $M(\bar{H}_n) = \prod_{\alpha=1}^{\infty} M(G'_{\alpha,n})$. We note that the set $\sum_{\alpha=1}^{\infty} G'_{m_\alpha,\alpha}$ where m_α is an arbitrary integer is open and contains $\bar{H}$. Furthermore, the manifold-measure of this open set contains the manifold $\sum_{\alpha=1}^{\infty} M(\bar{H}_\alpha)$. But the intersection of all manifolds arising from open sets of this type is equal to $\sum_{\alpha=1}^{\infty} M(\bar{H}_\alpha)$ [9]; this states that

$$M_{ex}(\bar{H}) \subseteqq \sum_{\alpha=1}^{\infty} M(\bar{H}_\alpha).$$

[9]) If we choose the integers m_n appropriately the projection of an arbitrary element f in $\mathfrak{H}$ on $M(G'_{m_n,n})$ will differ as little as we wish from its projection on $M(\bar{H}_n)$. Hence in turn, the projection of f on $\sum_{\alpha=1}^{\infty} M(G'_{m_\alpha,\alpha})$

Since $\prod_{\alpha=1}^{\infty} M(H_\alpha)$ is orthogonal to $\sum_{\alpha=1}^{\infty} M(\bar{H}_\alpha)$ the set H is measurable with manifold-measure $\prod_{\alpha=1}^{\infty} M(H_\alpha)$ [10]. The last statement of the theorem is established by considering complementary sets.

§ 3. We are ready to introduce the notion of the function of an operator A whose resolution is $E(\lambda)$. If $g(\lambda)$ is a real function the symbol $[g(\lambda) \leqq \mu]$ represents the set of points λ for which $g(\lambda) \leqq \mu$; for shortness, we denote it by H_μ. Let now $g(\lambda)$ be a real function defined and finite except on a set of zero measure with respect to $E(\lambda)$; let H_μ be measurable with respect to $E(\lambda)$, $-\infty < \mu < \infty$. Then the projections $E(H_\mu)$ constitute a resolution of the identity; for properties a), b) and c) in **§ 1** are satisfied. We say that $g(\lambda)$ is measurable with respect to $E(\lambda)$ and denote the transformation associated by means of (2) with the resolution $E(H_\mu)$ by the symbol $\int g(\lambda)\, dE(\lambda)$ or for short $g(A)$ [11]. Thus for us the equation $g(A) = \int g(\lambda) dE(\lambda)$ has but superficial significance. We note that $E(H_\mu)$ is commutative with any bounded transformation commutative with $E(\lambda)$. For as stated above, this is true of the operator $E(G)$ where G is any open set, hence by lemmas 2 and 1, for any measurable set H.

If $g(\lambda)$ is measurable, then it may be approximated uniformly except on a set of zero measure by measurable functions assuming only a finite or denumerable number of values having no finite

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will differ little from its projection on $\sum_{\alpha=1}^{\infty} M(\bar{H}_\alpha)$. This last statement may be established by first considering the equation

$$\|f - P_1 P_2 f\| = \|f - P_1 f + P_1 f - P_1 P_2 f\| \leqq$$
$$\leqq \|f - P_1 f\| + \|P_1(f - P_2 f)\| \leqq \|f - P_1 f\| + \|f - P_2 f\|$$

where P_1 and P_2 are commutative projections; we next consider the case of a finite number of such projections; then by lemma 1, we treat the infinite case.

[10]) Since $M_{ex}(H) + M_{ex}(\bar{H}) = \mathfrak{H}$, the equations $M_{ex}(H) \subseteqq M_1$, $M_{ex}(\bar{H}) \subseteqq M_2$, $M_1 \bullet M_2 = 0$ imply $M(H) = M_1$.

[11]) The notion of functions of an operator is also discussed by F. Maeda in a paper which has just reached us as we go to press, viz., Theory of Vector Valued Set Functions, *Journal of Science of the Hiroshima University*, Series A, 4 (1934), pp. 57–91. This author assumes the existence for Borel sets of a theory of measure such as we have carried out in **§ 2.** Then he defines integration of Baire functions with respect to a resolution of the identity and derives some of the properties of these integrals.

limiting values. The resolution corresponding to a function assuming only a denumerable number of values is constant except for at most a denumerable set of points. It is well known that the corresponding transformation has a pure point spectrum and hence is of a rather simple character. If $g(\lambda)$ is an arbitrary measurable function and $g_n(\lambda)$ are measurable functions of the special type just mentioned and approximating uniformly to $g(\lambda)$, then $g_n(A)$ converges to $g(A)$. For the argument presented in § 1 states that $g_n(A)$ is defined for an element f in $\mathfrak{H}$ if and only if $g(A)$ is defined for f. (We assume here that $|g(\lambda)-g_n(\lambda)|<M$ almost everywhere.) Furthermore the element $g_n(A)f$ is precisely one of the elements appearing in the converging sequence of elements which may be used to define $g(A)f$.

If A and B are bounded and commutative, polynomials in A and B possess the same property. The same is true of the operators corresponding to the limits in a well determined sense of these polynomials. We need not here discuss the manner in which these limits are determined; it suffices to say that among these "limit" operators we find the resolutions $E(\lambda)$ and $F(\mu)$ of A and B respectively[12]). Conversely, if the resolutions of two bounded operators are commutative, the operators themselves are commutative. This arises from the fact that the uniformly bounded operators which approximate to A and B are commutative (see (2)). In case the operators A and B are not both bounded, an unmodified equation of the type $AB=BA$ has little meaning[13]). This prompts us to introduce

Definition 2: *Two self-adjoint operators A and B are said to be commutative if their respective resolutions $E(\lambda)$ and $F(\mu)$ are commutative, $-\infty<\lambda,\ \mu<\infty$* [14]).

We are now in a position to state the principal properties of the operator $g(A)$. We have

Theorem 2: *Let A be self-adjoint and possess the resolution $E(\lambda)$. Let $g(\lambda)$, $g_1(\lambda)$ and $g_2(\lambda)$ be measurable with respect to $E(\lambda)$. Then*

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12) See [6]).

13) Problems arising in the definition of permutability of unbounded operators will be discussed by the author in a forthcoming publication.

14) The definition was introduced by J. Neumann, *Mathematische Grundlagen der Quantenmechanik* (Berlin, 1932), p. 90, and M. H. Stone, loc. cit., p. 301.

a) *$g(A)$ has the right-hand bound*[15]) *M if and only if $g(\lambda) \leqq M$ except for a set of zero measure with respect to $E(\lambda)$. An analoguous statement may be made for left-hand bounds.*
b) *$g(A)$ is commutative with any operator commutative with A.*
c) *$g(A)$ is a projection if and only if $g(\lambda) = 0$ or 1 everywhere except on a set of zero measure with respect to $E(\lambda)$.*
d) $$\int g_1(\lambda) dE(\lambda) + \int g_2(\lambda) dE(\lambda) = \int \{g_1(\lambda) + g_2(\lambda)\} dE(\lambda)$$
which is to be interpreted that the operator on the right is significant for an element f if each of the operators on the left is defined for f and then the equality is valid.
e) $$\left\{\int g_1(\lambda) dE(\lambda)\right\} \left\{\int g_2(\lambda) dE(\lambda)\right\} = \int g_1(\lambda) g_2(\lambda) dE(\lambda)$$
which is to be interpreted that the operator on the right is significant for an element f if the product of the operators on the left is defined for f and then the equality is valid.
f) *If $h(\lambda)$ is measurable with respect to the resolution $F(\lambda)$ of $g(A)$ then $h\{g(\lambda)\}$ is measurable with respect to $E(\lambda)$ and*
$$\int h(\lambda) dF(\lambda) = \int h\{g(\lambda)\} dE(\lambda).$$

Properties a) and c) are immediate consequences of our definition of $g(A)$. To establish b) we note that if an operator B is commutative with A, the resolution of B is commutative with the resolution of A and hence also with the resolution of $g(A)$.

In consideration of d) we see that our statement is true for all functions $g_1(\lambda)$ and $g_2(\lambda)$ which assume only a finite number of values. Hence the statement is true for any bounded measurable functions. Let the functions be unrestricted and let H_n be the measurable set for which both $|g_1(\lambda)| \leqq n$ and $|g_2(\lambda)| \leqq n$. Let the operator B_1 correspond to $g_1(\lambda)$, B_2 correspond to $g_2(\lambda)$ and C correspond to $g_1(\lambda) + g_2(\lambda)$. Then $B_1 E(H_n)$ corresponds to a function equal to $g_1(\lambda)$ when $|g_1(\lambda)| \leqq n$ and $|g_2(\lambda)| \leqq n$ and to zero otherwise; an analoguous statement may be made for $B_2 E(H_n)$. By what we have established for bounded operators,

[15]) M is said to be a right-hand bound of the transformation A with resolution $E(\lambda)$ if $E(M) = 1$. Similarly, m is a left-hand bound if $E(m) = 0$. These statements may also be expressed with the use of bilinear forms, viz., $(Af, f) \leqq M \|f\|^2$ or $(Af, f) \geqq m \|f\|^2$.

$$B_1 E(H_n) + B_2 E(H_n) = CE(H_n)$$

since $CE(H_n)$ corresponds to the sum of the functions just described. Now let f be an element in the domain of definition of B_1 and B_2; then $E(H_n) f \to f$ and in addition the sequences

$$B_1 E(H_n) f = E(H_n) B_1 f, \quad B_2 E(H_n) f = E(H_n) B_2 f$$

converge. Hence by the property of closure of a self-adjoint transformation mentioned in **§ 1** Cf is defined and $B_1 f + B_2 f = Cf$.

We note that e) is valid for two projection operators. Next, by d) it is valid for operators corresponding to measurable functions assuming only a finite number of values. Since any bounded measurable function may be approximated uniformly by functions of this simple type, e) is valid for bounded functions. Let the functions be unrestricted; let the operator C correspond to $g_1(\lambda) g_2(\lambda)$ and let B_1, B_2, and $E(H_n)$ have the meanings assigned to them above. We see that

$$\{B_1 E(H_n)\} \{B_2 E(H_n)\} = CE(H_n).$$

Let f be in the domain of B_2 and $B_2 f$ in that of B_1. Then

$$\{B_1 E(H_n)\} \{B_2 E(H_n)\} f = B_1 E(H_n) B_2 f = E(H_n) B_1 B_2 f.$$

This sequence of elements converges with $1/n$ hence property e) is established. We point out that the demonstration of the last two properties involves a method tantamount to an integration with respect to what might be called a resolution of the identity in two dimensions.

In order to establish f) we see first of all that the measure of an open interval with respect to $F(\lambda)$ is precisely the measure with respect to $E(\lambda)$ of the set of points for which $g(\lambda)$ takes values in this open interval; the statement allows of an immediate extension to arbitrary open sets. Now let Y be any set measurable with respect to $F(\lambda)$; let X be the set of all points λ such that $g(\lambda)$ is a point in Y. We note in passing that certain points in Y may have no correspondents in X. We shall prove that X is measurable with respect to $E(\lambda)$ and indeed that its measure with respect to $E(\lambda)$ is identical with the measure of Y with respect to $F(\lambda)$. The measure of Y with respect to $F(\lambda)$ may be expressed as the intersection of the measures of a denumerable number of sets each containing Y (by lemma 2). We denote the intersection of these open sets by Y_1 and note that $Y_1 \supseteqq Y$. By the first statement of this paragraph we see that the measure of Y with respect to $F(\lambda)$ is equal to the intersection of the measures of a denumerable number of sets measurable with respect to $E(\lambda)$ each of them containing X. The intersection of these sets measurable with respect to $E(\lambda)$ is denoted by X_1; we have $X_1 \supseteqq X$. Either by considering the intersection of open sets containing the complement of Y or the sum of closed sets contained in Y, we obtain a set $Y_2 \subseteqq Y$ whose measure with respect to $F(\lambda)$ is identical with that of Y. The corresponding set X_2 is contained in X, $X_2 \subseteqq X$, and has a measure with respect to $E(\lambda)$ equal to that of Y with respect to $F(\lambda)$ hence equal to that of X_1. This means that X is measurable and that its measure with respect to $E(\lambda)$ is identical with the measure of Y with respect to $F(\lambda)$.

Now let $h(\lambda)$ be any function measurable with respect to $F(\lambda)$. We have just proved that the measure with respect to $F(\lambda)$ of the set $[h(\lambda) \leqq c]$ is identical with the measure with respect to $E(\lambda)$ of the set $[h\{g(\lambda)\} \leqq c]$. Hence

$$\int h(\lambda)\, dF(\lambda) = \int h\{g(\lambda)\}\, dE(\lambda).$$

We have for the sake of simplicity restricted ourselves to real functions of an operator. If we let $g(\lambda) = g_1(\lambda) + i g_2(\lambda)$ where $g_1(\lambda)$ and $g_2(\lambda)$ are real and measurable with respect to $E(\lambda)$, $g(A)$ represents a normal rather than a self-adjoint operator. The theory of such operators can be carried out along the lines developed above.

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