Bibliographie.

The late Raymond E. A. C. Paley and Norbert Wiener, Fourier Transforms in the Complex Domain (American Math. Society Colloquium Publications, Volume XIX), VIII + 184 pages, New York, American Mathematical Society, 1934.

One of the most characteristic features of this book is that there are discussed in it the most varied topics of analysis with the powerful aid of the Fourier—Mellin transforms. As an exhaustive exposition of all results contained in this remarkable and original work is impossible for lack of space, the reviewer has to content himself with an enumeration of the main problems dealt with.

Chapter I gives a new proof of Carleman's theorem on quasi-analytic classes based upon a theorem on the Fourier transforms of functions belonging to the class $L_2(-\infty, \infty)$ and vanishing on a half-axis. Chapter II proves Szász's conditions relating to the closure in $L_2(0,1)$ of sets $\{x^{\lambda_n}\}$ with complex λ_n and the Müntz—Szász conditions for the possibility of uniform approximation of continuous functions with such powers.

The next two chapters are devoted to Watson transforms and to some integral equations, principally to the homogeneous integral equation on $(0,\infty)$ with kernels of the type $K(x-y)$, vanishing exponentially for large values $|x|$.

The following chapters give some fine theorems on the distribution of zeros for entire functions of the exponential type, with applications to the problem of characterising the sets of real numbers λ_n for which the set $\{e^{i\lambda_n x}\}$ is closed in $L_2(a, b)$. In order to establish theorems on non-harmonic Fourier series, the authors develop a theorem concerning a set $\{g_n\}$ of elements in Hilbert space which differs but slightly from a closed orthonormal set $\{f_n\}$, i. e.

$$\left\| \sum_n a_n (f_n - g_n) \right\| \leqq \vartheta \left(\sum_n |a_n|^2 \right)^{1/2}, \qquad \vartheta < 1,$$

the a_n being arbitrary complex numbers for which the right-hand side sum converges. It may be pointed out that this theorem could be derived also from the following one: *Let A be a linear transformation in Hilbert space for which*

$$\|Ah\| \leqq M\|h\|, \qquad M < 1$$

(h being an arbitrary element of Hilbert space), then $(E-A)^{-1}$ exists and we have

$$(1+M)^{-1}\|h\| \leqq \|(E-A)^{-1}h\| \leqq (1-M)^{-1}\|h\|$$

(*cf.* for instance F. Riesz, *Les systèmes d'équations linéaires à une infinité d'inconnues* (Paris, 1913), p. 92).

We find then some much interesting theorems on non-harmonic

Fourier series, gap theorems and an extension of Wiener's generalised harmonic analysis for complex arguments.

The last two chapters deal with random functions and their harmonic analysis using the theory of integration in a space of infinite dimensions as well as with applications to ergodic problems and to the theory of Brownian motion.

The book is written very clearly and in a suggestive manner, the demonstrations are all carried out in their details. One is feeling here a complete harmony between the classical theory of analytical functions and the most modern branches of the theory of real functions.

The book is headed by a picture of the late R. E. A. C. PALEY reminding us of the great loss to mathematical sciences by his early death.

The surviving author dedicates the book to Professors HARDY and LITTLEWOOD, teachers of both authors.

Béla de Sz. Nagy.

Willard van Orman Quine, A System of Logistic, with a foreword by A. N. WHITEHEAD, XI + 204 pages, Cambridge Massachusetts, Harvard University Press, 1934.

The system of symbolic logic developed in the present book is closely related to that of Whitehead's and Russell's *Principia Mathematica* (*PM*). However, Quine's system is a somewhat more general one. Indeed, in *PM*, classes and dyadic relations had to be treated separately; the corresponding theorems on triadic and tetradic relations and so on, could be proved by similar methods. However, it is impossible to demonstrate or even to express, in the system of *PM*, any theorem dealing with relations in general, i. e. with n-adic relations without specifying n. This inconvenience is avoided in Quine's system by regarding the dyad (i. e. ordered couple) x, y of terms of whatever type x and y as a term of a new type again. By iteration we obtain triads, tetrads and so on; for instance, the triad x, y, z is identified with the dyad $x, (y, z)$ formed of the term x and the dyad y, z. In this way, triads become a special kind of dyads, tetrads that of triads and so on, as dyads itselves are to be regarded a special kind of terms. An an n-adic relation can be considered as a class of n-ads, the theory of relations in general becomes not only possible, but even a special case of the theory of classes.

In Quine's system, the standard form of a proposition is "x is a member of the class α"; the proposition itself is identified with the dyad α, x formed of the "predicate" α and the "subject" x. Propositions are, therefore, terms of special types, viz. of the type of a dyad whose first element has the type of a class capable to contain or not an element of the same type as the second element of the dyad. Propositional functions are not used but are replaced by their extensions; therefore, the so-called non-ramified type theory can serve as basis. The type-range of a variable, or rather the relative restrictions on the type-ranges of the variables occurring in an expression required to render it significant, is provided by the context rather than by the shape of the variables. However, for the sake of convenience, special propositional and class variables are used.

Besides the binary operation of "ordination", symbolized by the comma, there are needed but two primitive ideas, the unary operation of "congeneration" and the variable-binding operation of "abstraction". The first one has to operate on a class α and produces the class $[\alpha]$ of all those classes in which α is included. The second one operates on a propositional expression "- - -" containing some or no free occurrence of a variable "x" and generates the class $\hat{x}$ (- - -) of all terms x such that - - - holds. Particularly, if "x" does not occur in "- - -", $\hat{x}$ (- - -) denotes the universal class or the null class according as - - - expresses a true or a false proposition. Thus, in this case, $\hat{x}$ (- - -) can be identified with the truth-value of the proposition - - -; this device is used to define the operations of the propositional calculus in terms of the congeneration and abstraction. For instance, material implication $p \supset q$ ("p implies q") is defined as $[\hat{x}p]$, $\hat{y}q$. The abstraction is the only way of binding a variable; for instance the universal quantifier, "for every term x, - - - holds", is expressed as U, $\hat{x}$ (- - -), where U, the class whose only element is the universal class, is of course defined equally by means of abstraction and congeneration.

As informal rules of inference are adopted, besides the familiar rule of substitution and that of the inference of the consequent ("*modus ponens*"), two rules called of subsumption and of concretion, the former of which enables us to bind a free variable, occurring in a theorem or postulate, by means of the universal quantifier; the latter provides the equivalence of the proposition $\hat{x}$ (- - -), y to the result of substituting "y" for "x" in "- - -".

The system is based upon six formal postulates. Some of them have a rather forbidding aspect; but one could hardly avoid a complicated form of postulates in a system with a so restriced number of primitive ideas among which the operations of propositional calculus do not occur. However, in order to make the reader familiar with the postulates, the author discusses in detail their intuitive meaning and validity.

On the basis of the postulates the propositional calculus, the theory of identity, of universality and existence, the fundamental notions of class and relation calculus etc. are developed, shortly all theories which precede the cardinal arithmetic in *P M*. Finally, the adequacy of the system to that of *PM* is proved. For abbreviation of formal proofs a scheme due to ŁUKASIEWICZ is adopted and fitted to the system. For each formally derived theorem, not merely lemma for subsequent theorems, the intuitive meaning is carefully explained. This appears also at a glance on the text; indeed, the ratio of English text to formulas is much greater than that in *PM*. The fact that the author could develop, in spite of this better ratio, within two hundred pages a theory equivalent to that on the first four hundred pages of *PM*, makes evident the advantages of his system.

L. Kalmár.

Max Deuring, Algebren (Ergebnisse der Math. und ihrer Grenzgebiete, vierter Band, Heft 1), V + 143 S., Berlin, J. Springer, 1935.

Die neue Entwicklung der Theorie der Algebren (nichtkommutativen Ringen über einem Körper mit einer kommutativen „skaleren" Multiplikation für die Elemente des Körpers und des Ringes) hat das Erscheinen dieses Berichtes notwendig gemacht. Nach den Grundlagen in I setzt Verfasser in II die Struktursätze, in der allgemeinsten bisher bekannten Fassung, in helles Licht. Das wichtigste methodische Hilfsmittel ist weiterhin der Begriff der Darstellung in Noetherscher Begründung. In III folgt ein Überblick über die Theorie der Darstellungen und Darstellungsmoduln sowie Darstellungen der Algebren, in IV die Theorie der einfachen Algebren (nichtkommutative Galoissche Theorie, Theorie der Zerfällungskörper) und in V die Theorie der Brauer—Noetherschen Faktorensysteme. Die zweite Hälfte des Buches behandelt die Arithmetik der Algebren. Nach der Theorie der ganzen Größen in VI (Idealtheorie der Algebren, die Idealklassentheorie und p-adische Erweiterungen von Algebren), die eine starke Vereinfachung gegenüber der bisherigen Begründung bedeutet, entwickelt Verfasser in VII die tiefere Theorie der Algebren über einen Zahlkörper. HASSE hat die Zusammenhänge zwischen den Struktureigenschaften der Algebren und der Arithmetik der Zahlkörper, insbesondere der Klassenkörpertheorie und des Reziprozitätsgesetzes erkannt. Dementsprechend gibt Verfasser in VII einige Anwendungen der Algebrentheorie, deren wichtigste der Hassesche Beweis des Artinschen Reziprozitätsgesetzes und die Noethersche Verallgemeinerung des Hauptgeschlechtsatzes sind.

Verfasser gibt eine vollständige Darstellung der Theorie der Algebren, mit Hinweisen auf die bezügliche Literatur, so daß dieser Bericht zugleich ein vollständiges Lehrbuch ist, das demjenigen, der in diese elegante Theorie eindringen will, unentbehrlich ist.

L. Zányi.

Antoni Zygmund, Trigonometrical Series (Monografje Matematyczne, Tom V), IV + 331 pages, Warszawa—Lwów, 1935.

This volume of the *Monografje Matematyczne* contains the standard results of the theory of trigonometrical series as well as the newest ones. As a matter of fact, it is chiefly dealing with Fourier series. The author treats exhaustively his topic so vastly enriched by recent investigations, not yet inserted in the textbooks.

The abundance and the diversity of its subject-matter makes impossible even the bare enumeration of its problems and so we are restricted only to mention the most important ones.

Chapter I and II present beside the fundamental facts concerning the Fourier series and its coefficients the different convergence criteria together with their connections. Chapter III is dealing with the problems of summability of the Fourier series (theorems of FEJÉR and of M. RIESZ about the summability (C, r); Abel summability). The generalisation of the summability theorem of FEJÉR is the subject also of one part of Chapter X. concerned with the convergence of the sequence

$$\frac{1}{n+1} \sum_{\nu=0}^{n} |S_\nu(x) - f(x)|^k$$

(CARLEMAN, HARDY, LITTLEWOOD, ZYGMUND); we find in this chapter the most recent investigations of HARDY and LITTLEWOOD about the Cesàro and Abel summation of the Fourier series.

Chapter IV deals with the theorem of RIESZ—FISCHER and with the problem of strong convergence; it presents here the necessary and sufficient conditions for belonging to the different classes of functions as far as they are to be expressed by Fourier series (theorems of YOUNG, ZYGMUND and STEINHAUS). Chapter IX is attached to the same circle of thougt; it presents the theorem of HAUSDORFF—YOUNG, its generalisation by F. RIESZ and the respective theorems of PALEY and HARDY—LITTLEWOOD as well as two theorems concerning lacunary series, special cases of certain theorems of BANACH dealing with orthogonal series. It may be observed that the second of these theorems, for the trigonometrical case, was found by SIDON independently and before Banach's paper; indeed, Sidon's one dealing with the subject, though not printed before 1932, was accepted for publication in 1928. Also Chapter VII is occupied with the strong convergence of the Fourier series and of the conjugate series (theorems of M. RIESZ, FEJÉR and ZYGMUND); here we find the investigations of HARDY, LITTLEWOOD, F. and M. RIESZ about conjugate series.

Chapter V is concerned about the series with special coefficients whereas Chapter VI contains the results about the absolute convergence of trigonometrical series (LUSIN, DENJOY, FATOU, S. BERNSTEIN, O. SZÁSZ, ZYGMUND, SIDON and N. WIENER).

Chapter XIII is dealing with the divergence-phenomena (DU BOIS REYMOND, FEJÉR, KOLMOGOROFF); besides we find here the theorems of ROGOSINSKI and CRAMÉR together with an account of the phenomenon of GIBBS.

The two last chapters develop Riemann's theory of trigonometrical series together with a concise discussion of Fourier's integrals.

Finally, it is to be mentioned that at the end of every chapter, excellent and very judiciously chosen problems lead the reader into the most delicate details of the theory. In spite of the conciseness, the argumentation is always clear. In summary, Professor Zygmund's book is a rich, useful and most delightful work.

G. Grünwald.