On Orthocentric Simplexes.

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A simplex $[P_1P_2\ldots P_{n+1}]$ of the n-dimensional space is said to be orthocentric if its heights, i. e. the perpendiculars from its vertices P_ν to the opposite faces $[P_1P_2\ldots P_{\nu-1}P_{\nu+1}\ldots P_{n+1}]$ are concurrent, their meeting point P_0 being called the orthocentre of the simplex.

For some purposes it is more convenient to consider the set of $n+2$ points $P_0P_1\ldots P_{n+1}$ as a whole, called an orthocentric set of $n+2$ points in n-dimensional space, each point of the set being the orthocentre of the simplex formed by the others.

Orthocentric simplexes and sets of points had been investigated by MEHMKE, RICHMOND, LOB[1]).

The first object of the present paper is to establish the following characteristic property of orthocentric sets of points (§. 1):

If the set $P_0P_1\ldots P_{n+1}$ of $n+2$ points in the n-dimensional space is orthocentric, then and only then can the mutual distances $\overline{P_iP_j}$ be expressed by $n+2$ symmetric parameters λ_i in the form:

$$\overline{P_iP_j}^2=\lambda_i+\lambda_j \qquad (i,j=0,1,\ldots,n+1;\ i\neq j) \tag{1}$$

the parameters λ_i being restricted only by the relations[2])

$$\frac{1}{\lambda_0}+\frac{1}{\lambda_1}+\ldots+\frac{1}{\lambda_{n+1}}=0, \qquad \lambda_i+\lambda_j>0 \text{ for } i\neq j. \tag{2}$$

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[1]) R. MEHMKE, Ausdehnung einiger elementarer Sätze über das ebene Dreieck auf Räume von beliebig viel Dimensionen, *Archiv für Math. und Phys.*, **70** (1884), pp. 210—218; H. W. RICHMOND, On extensions of the property of the orthocentre, *Quarterly Journal*, **32** (1901), pp. 251—255; H. LOB, The Orthocentric Simplex in Space of Three and Higher Dimensions, *Math. Gazette*, **19** (1935), p. 102—108.

[2]) The equations (1) and (2) are in connexion to a problem proposed by

This result includes obviously the determination of an orthocentric simplex in n dimensions by means of $n+1$ symmetric, independent[3]) parameters.

The n-dimensional measure V of the orthocentric simplex $[P_1 P_2 \dots P_{n+1}]$ will be shown to be given by

$$(3) \qquad n!\,V = \left\{\lambda_1 \lambda_2 \dots \lambda_{n+1}\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\dots+\frac{1}{\lambda_{n+1}}\right)\right\}^{\frac{1}{2}} = \frac{\sqrt{-\prod\limits_{\nu=0}^{n+1}\lambda_\nu}}{\lambda_0}$$

and from this result it follows immediately that if the masses $\frac{1}{\lambda_\nu}$ are placed resp. at the points P_ν, $(\nu = 0, 1, \dots, n+1)$, then each point is the barycentre of the masses placed at the others. According to (2) there is one and only one of negative value amongst the parameters λ_i, consequently an orthocentric set of points contains always one and only one point, which is interior[4]) to the simplex formed by the others.

In §. 2, we are going to establish the following connexion between orthocentric point-systems and orthogonal matrices:

The $n+1$ *points* $P_i(\varepsilon_{i1}, \varepsilon_{i2}, \dots, \varepsilon_{i,n+1})$, $(i = 1, 2, \dots, n+1)$ *in the n-dimensional space, whose homogenous cartesian coordinates*

W. H. Laverty, *Mathematical Questions with Solutions from the Educational Times*, **7** (1867), pp. 99—102; see also *The Collected Math. Papers* of A. Cayley, Vol. VII (Cambridge. 1894), pp. 578—581. For $n=3$ the equation (2) has been established by O. Dziobek, Über eine Erweiterung des Gauss'-schen Pentagramma mirificium auf ein beliebiges sphärisches Dreieck, *Archiv für Math. und Phys.* (2) **16** (1898) pp. 320—326.

[3]) For the case $n=3$ see the author's paper, A magasságponttal biró tetraéderről, *Math. és Fiz. Lapok*, **45** (1938), pp. 18—35. The parameters λ_i are closely related to the problem of determinig an n-dimensional simplex of maximal volume, the volumes of its faces being given. The maximal simplex must be orthocentric, and the multipliers which enter in the discussion of the problem, are identical to the parameters λ_i introduced above. See C. W. Borchardt, Über die Aufgabe des Maximum, welche der Bestimmung des Tetraeders von grösstem Volumen bei gegebenem Flächeninhalt der Seitenflächen für mehr als drei Dimensionen entspricht, *Math. Abhandlungen der Akademie Berlin*, 1866, pp. 121—155; *Gesammelte Werke*, (Berlin, 1888), pp. 201—232 and L. Kronecker, Zur algebraischen Theorie der quadratischen Formen, *Monatsberichte der Akademie Berlin*, 1872, pp. 490—504; *Gesammelte Werke*, Bd. I. (Leipzig, 1895), pp. 283—301.

[4]) In the limiting case, when two of the parameters e. g. λ_1 and λ_2 vanish, the points P_1 and P_2 coincide, and there is no interior point.

ε_{ij} are the elements of an orthogonal matrix, form together with the origin an orthocentric set of $n+2$ points. And conversely, if the "interior" point of an orthocentric set will be placed at the origin, then an orthogonal matrix ε_{ij} can be found, by means of which the coordinates of the other points are expressed in the form:

$$(3) \qquad x_{ij} = \varrho \frac{\varepsilon_{ij}}{\varepsilon_{i,n+1}} \quad (i=1,2,\ldots,n+1;\ j=1,2,\ldots,n).$$

The parameters λ_i defined in §. 1 are now given by

$$(4) \qquad \lambda_0 = -\varrho^2;\ \lambda_i = \frac{\varrho^2}{\varepsilon^2_{i,n+1}} \quad (i=1,2,\ldots,n+1)$$

while the other elements ε_{ij} determine the position of the set relative to the axes.

An immediate consequence of this result is the following theorem: A necessary and sufficient condition that a set of points should be orthocentric is that it should be possible to arrange masses at the points in such a manner as to form a system, the principal inertial quadric of which is a hypersphere.

As an extension of the notion of the FEUERBACH circle it has been shown by the authors mentioned above that α) the middle-points of the "edges", β) the orthocentres and barycentres of the "faces" are situated on a hypersphere. In §. 3, we propose to establish the existence of the following set of $n-1$ hyperspheres of FEUERBACH (the first and last of which was discovered by RICHMOND, resp. MEHMKE):

The orthocentres[5]) and the barycentres of all the $k-1$-dimensional simplexes of an orthocentric simplex of n dimensions (specified by the parameters $\lambda_1,\ldots,\lambda_{n+1}$) belong to a hypersphere S_{k-1} of radius

$$(4) \qquad \frac{1}{2k}\sqrt{(n+1-2k)^2\lambda_0+\lambda_1+\lambda_2+\ldots+\lambda_{n+1}} \qquad (k=1,2,\ldots,n).$$

The middle points C_{k-1} of these hyperspheres S_{k-1} are lying on the "Euler line" which joins the orthocentre P_0 and the barycentre G of the simplex, and divide the distance $\overline{GP_0}$ at the ratio:

$$(4') \qquad \frac{\overline{GC_{k-1}}}{\overline{P_0C_{k-1}}} = \frac{n+1-2k}{n+1} \qquad (k=1,2,\ldots,n).$$

[5]) The orthocentre of an edge ($k=2$) is obviously the orthogonal projection of the orthocentre of the simplex on this edge. In the case $k=1$ we get from (4) the radius of the circumscribed sphere.

§. 1.

If the simplex $[P_1 \ldots P_{n+1}]$ in the n-dimensional space is orthocentric and its orthocentre is P_0, then by definition:

$\overline{P_0P_i} \perp$ hyperplane $\overline{P_1P_2 \ldots P_{i-1}P_{i+1} \ldots P_{n+1}}$ $\quad (i = 1, 2, \ldots, n+1)$
therefore

$$\overline{P_0P_i} \perp \overline{P_kP_l} \perp \overline{P_0P_j} \quad (i, j \neq k, l)$$

and

$$\overline{P_kP_l} \perp \text{ plane } \overline{P_0P_iP_j}.$$

Hence $\overline{P_iP_j} \perp \overline{P_kP_l}$, i. e. the set of $n+2$ points $P_0P_1 \ldots P_{n+1}$ is an orthocentric set of points, each point P_i of the set being the orthocentre of the simplex $[P_0P_1 \ldots P_{i-1}P_{i+1} \ldots P_{n+1}]$ formed by the others.

Denoting $\overline{P_iP_j} = \varrho_{ij}$, we have[6]) from $\overline{P_iP_j} \perp \overline{P_kP_l}$

$$\left.\begin{array}{ll} (5) & \varrho_{ik}^2 + \varrho_{jl}^2 = \varrho_{il}^2 + \varrho_{jk}^2 \\ \text{or} & \\ (6) & \varrho_{ij}^2 + \varrho_{ik}^2 - \varrho_{jk}^2 = \varrho_{ij}^2 + \varrho_{il}^2 - \varrho_{jl}^2 \end{array}\right\} (i, j, k, l = 0, 1, \ldots, n+1).$$

It is obvious by (6), that $\varrho_{ij}^2 + \varrho_{ik}^2 - \varrho_{jk}^2$ is independent of j and k, consequently the quantities

$$(7) \qquad \lambda_i = \varrho_{ij}\varrho_{ik} \cos P_j\widehat{P_i}P_k = \frac{\varrho_{ij}^2 + \varrho_{ik}^2 - \varrho_{jk}^2}{2}$$

may be introduced, as symmetric parameters.

The expression of the distances $\overline{P_iP_j}$ in terms of the λ_i is then

$$(1) \qquad \overline{P_iP_j}^2 = \varrho_{ij}^2 = \lambda_i + \lambda_j.$$

Applying the well-known formula of the measure of a $k-1$-dimensional simplex,

$$(3') \quad -(-2)^{k-1}(k-1)!^2\, V[P_{\nu_1}, P_{\nu_2}, \ldots, P_{\nu_k}]^2 =$$

$$= \begin{vmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & \varrho_{\nu_1\nu_2}^2 & \ldots & \varrho_{\nu_1\nu_k}^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & \varrho_{\nu_k\nu_1}^2 & \varrho_{\nu_k\nu_2}^2 & \ldots & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & \lambda_{\nu_1}+\lambda_{\nu_2} & \ldots & \lambda_{\nu_1}+\lambda_{\nu_k} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & \lambda_{\nu_k}+\lambda_{\nu_1} & \lambda_{\nu_k}+\lambda_{\nu_2} & \ldots & 0 \end{vmatrix},$$

we get immediately that the measure of the $k-1$-dimensional simplex $[P_{\nu_1}P_{\nu_2} \ldots P_{\nu_k}]$ (which is obviously orthocentric as well

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[6]) A necessary and sufficient condition that the diagonals $\overline{P_iP_j}$ and $\overline{P_kP_l}$ of a quadrangle $P_iP_kP_jP_l$ should be orthogonal, is that its sides satisfy the equation (5).

as $[P_1 \dots P_{n+1}]$) is given by

$$(k-1)!^2 V_{k-1}[P_{\nu_1} \dots P_{\nu_k}]^2 = \lambda_{\nu_1}\lambda_{\nu_2}\dots\lambda_{\nu_k}\left(\frac{1}{\lambda_{\nu_1}}+\frac{1}{\lambda_{\nu_2}}+\dots+\frac{1}{\lambda_{\nu_k}}\right).$$

$P_0, P_1, \dots, P_{n+1}$ are points of a space of n dimensions, therefore $V_{n+2}=0$, consequently

$$\frac{1}{\lambda_0}+\frac{1}{\lambda_1}+\dots+\frac{1}{\lambda_{n+1}}=0. \tag{2}$$

This relation (2) together with (1) shows clearly that amongst the parameters λ_i there is always exactly one which is negative, the others being positive. And vice versa, every set of real values $\lambda_0, \lambda_1, \dots, \lambda_{n+1}$, which satisfies (2) and where only one λ_i is negative, determines a real orthocentric point system.

Suppose $\lambda_0<0$, $\lambda_1, \lambda_2, \dots, \lambda_{n+1}>0$, then denoting $-\lambda_0\lambda_1\dots\lambda_{n+1}$ by Λ^2, we have

$$n!\,V^{(i)} = n!\,V[P_0P_1\dots P_{i-1}P_{i+1}\dots P_{n+1}] = \frac{\Lambda}{\lambda_i}$$

and

$$V^{(i)} : V^{(j)} = \frac{1}{\lambda_i} : \frac{1}{\lambda_j} \tag{8}$$

which relation shows clearly, that the barycentric coordinates of P_i relative to the fundamental simplex $[P_0P_1\dots P_{i-1}P_{i+1}\dots P_{n+1}]$ are

$$\frac{1}{\lambda_0}, \frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_{i-1}}, \frac{1}{\lambda_{i+1}}, \dots, \frac{1}{\lambda_{n+1}}.$$

§. 2.

Consider the set of $n+1$ points P_i in the n-dimensional space whose rectangular coordinates are given by

$$(9)\quad \begin{array}{ll} (P_1) & x_{11} : x_{12} : \dots : x_{1n} : 1 = \varepsilon_{11} : \varepsilon_{12} : \dots : \varepsilon_{1,n+1} \\ (P_2) & x_{21} : x_{22} : \dots : x_{2n} : 1 = \varepsilon_{21} : \varepsilon_{22} : \dots : \varepsilon_{2,n+1} \\ \dots & \dots \\ (P_{n+1}) & x_{n+1,1} : x_{n+1,2} : \dots : x_{n+1,n} : 1 = \varepsilon_{n+1,1} : \varepsilon_{n+1,2} : \dots : \varepsilon_{n+1,n+1} \end{array}$$

and suppose that the matrix $\|\varepsilon_{ij}\|$ is orthogonal, i. e.

$$\sum_{\nu=1}^{n+1} \varepsilon_{i\nu}\varepsilon_{j\nu}=0;\ \sum_{\nu=1}^{n+1}\varepsilon_{i\nu}^2=1 \qquad (i,j=1,2,\dots,n+1;\ i\neq j).$$

We shall prove that these points $P_1, P_2, \dots, P_{n+1}$ and the origin:

$$(9')\qquad (P_0)\quad x_{01} : x_{02} : \dots : x_{0n} : 1 = 0 : 0 : \dots : 1$$

form an orthocentric set.

Starting from the values of x_{ij} given in (9), (9'), we have

$$\sum_{\nu=1}^{n} x_{i\nu} x_{j\nu} = \frac{\sum_{\nu=1}^{n} \varepsilon_{i\nu}\varepsilon_{j\nu}}{\varepsilon_{i,n+1}\varepsilon_{j,n+1}} = -1 \quad (i \neq j;\ i, j = 1, 2, \ldots, n+1)$$

or

$$(10) \qquad \sum_{\nu=1}^{n} (x_{i\nu} - x_{k\nu})(x_{j\nu} - x_{l\nu}) = 0 \quad (i, j, k, l = 0, 1, 2, \ldots, n+1)$$

consequently $\overline{P_i P_k} \perp \overline{P_j P_l}$ and the orthocentric property of the set $P_0 P_1 \ldots P_{n+1}$ is thus proved.

We may express the distances $\overline{P_i P_j}$ by means of the ε_{ij}:

$$\left.\begin{aligned} \overline{P_0 P_j}^2 &= \sum_{\nu=1}^{n} x_{j\nu}^2 = \sum_{\nu=1}^{n} \left(\frac{\varepsilon_{j\nu}}{\varepsilon_{j,n+1}}\right)^2 = \frac{1}{\varepsilon_{j,n+1}^2} - 1 \\ \overline{P_i P_j}^2 &= \sum_{\nu=1}^{n} (x_{i\nu} - x_{j\nu})^2 = \\ &= \sum_{\nu=1}^{n} \left(\frac{\varepsilon_{i\nu}}{\varepsilon_{i,n+1}} - \frac{\varepsilon_{j\nu}}{\varepsilon_{j\,n+1}}\right)^2 = \frac{1}{\varepsilon_{i,n+1}^2} + \frac{1}{\varepsilon_{j,n+1}^2} \end{aligned}\right\} (i, j = 1, 2, \ldots, n+1).$$

Comparing these expressions with those in §. 2, we see immediately that, if the unity of length is conveniently chosen, the parameters

$$\lambda_0, \lambda_1, \ldots, \lambda_{n+1}$$

can be identified with the quantities

$$-1, \frac{1}{\varepsilon_{1,n+1}^2}, \ldots, \frac{1}{\varepsilon_{n+1,n+1}^2}.$$

The conditions of orthogonality

$$\sum_{i=1}^{n+1} \varepsilon_{i,n+1}^2 x_{i\nu} = \sum_{i=1}^{n+1} \varepsilon_{i,n+1}\varepsilon_{i\nu} = 0 \quad (\nu = 1, 2, \ldots, n)$$

show moreover, that the origin P_0, viz. the point corresponding to the negative parameter is an interior point of the simplex formed by the others.

By what precedes, it is seen at once that the coordinates of any orthocentric set of points can be expressed by means of an orthogonal matrix. Suppose indeed, that an orthocentric set of $n+2$ points $P_0 P_1 \ldots P_{n+1}$ in the n-dimensional space be specified by the parameters

$$\lambda_0 < 0,\ \lambda_1, \lambda_2, \ldots, \lambda_{n+1} > 0$$

and determine an orthogonal matrix $\|\varepsilon_{ij}\|$ such that the elements of its last column satisfy the conditions

$$\varepsilon_{1,n+1}^2 = -\frac{\lambda_0}{\lambda_1};\ \varepsilon_{2,n+1}^2 = -\frac{\lambda_0}{\lambda_2};\ \ldots;\ \varepsilon_{n+1,n+1}^2 = -\frac{\lambda_0}{\lambda_{n+1}},$$

the other elements being arbitrary.

Then, taking P_0 as origin, the coordinates of the points of the orthocentric set are

$$x_{0\nu} = 0;\ x_{i\nu} = \sqrt{-\lambda_0}\,\frac{\varepsilon_{i\nu}}{\varepsilon_{i,n+1}} \quad (\nu = 1, 2, \ldots, n;\ i = 1, 2, \ldots, n+1)$$

the position of the set relative to the axes being determined by the arbitrary elements of the matrix $\|\varepsilon_{ij}\|$.

§. 3.

If a set of points $P_1, P_2, \ldots, P_r$ is determined by their mutual distances $\overline{P_iP_j} = \varrho_{ij}$, and if at the points P_i masses α_i, resp. β_i are placed, then the distance of the barycentres $A(\alpha_1, \ldots, \alpha_r)$ and $B(\beta_1, \ldots, \beta_r)$ of these two systems of masses is given by[7])

$$\overline{AB}^2 = -\sum_{i=1}^{r}\sum_{j=1}^{r} \varrho_{ij}^2(\alpha_i - \beta_i)(\alpha_j - \beta_j)$$

provided that

$$\sum_{i=1}^{r}\alpha_i = \sum_{i=1}^{r}\beta_i = 1.$$

If the points $P_1, P_2, \ldots, P_{n+1}$ are the vertices of an orthocentric simplex in n dimensions, then substituting $\varrho_{ij}^2 = \lambda_i + \lambda_j$, we get immediately the following expression for the distance of two points A and B, whose barycentric coordinates relative to the orthocentric simplex $[P_1 \ldots P_{n+1}]$ are α_i, resp. β_i

$$\overline{AB}^2 = \sum_{i=1}^{n+1}\lambda_i(\alpha_i - \beta_i)^2;\ \Sigma\alpha_i = \Sigma\beta_i = 1. \tag{11}$$

In order to find the analogues of the FEUERBACH circle we are going to investigate, by means of the formula (11), whether there is a point on the line joining the barycentre G and the orthocentre P_0 of the simplex $[P_1 \ldots P_{n+1}]$ which is equidistant from the barycentres of all the $k-1$-dimensional simplexes contained in $[P_1 \ldots P_{n+1}]$.

7) See e. g. E. CESÀRO—G. KOWALEWSKY, *Vorlesungen über natürliche Geometrie* (Leipzig, 1901), pp. 297—298.

The barycentric coordinates of G, resp. P_0 being $1, 1, \ldots, 1$, resp. $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_{n+1}}$, the barycentric coordinates of the points C of the line joining them are

(12) $$\alpha_\nu = 1 + \frac{\gamma}{\lambda_\nu} \quad (\nu = 1, 2, \ldots, n+1), \ \alpha = \Sigma \alpha_\nu = n + 1 - \frac{\gamma}{\lambda_0}$$

where γ denotes a variable parameter.

The barycentric coordinates of the barycentre G_{k-1} of the $k-1$-dimensional simplex $[P_1 P_2 \ldots P_k]$ may be obviously taken to be:

(13) $$\beta_1 = \beta_2 = \ldots = \beta_k = \frac{1}{k}; \ \beta_{k+1} = \ldots = \beta_{n+1} = 0.$$

Hence

$$\overline{CG_{k-1}}^2 = \frac{\sum_{i=1}^{k} \left[\frac{\alpha}{k} - \left(1 + \frac{\gamma}{\lambda_i}\right)\right]^2 \lambda_i + \sum_{j=k+1}^{n+1} \left(1 + \frac{\gamma}{\lambda_j}\right)^2 \lambda_j}{\alpha^2} =$$

$$= \frac{\sum_{i=1}^{k} \frac{\alpha}{k} \left[\left(\frac{\alpha}{k} - 2\right)\lambda_i - 2\gamma\right] + \sum_{\nu=1}^{n+1} \left(1 + \frac{\gamma}{\lambda_\nu}\right)^2 \lambda_\nu}{\alpha^2}.$$

This expression will be independent of the choice of the vertices $P_1, P_2, \ldots, P_k$, if $\alpha = 2k$ and $\gamma_{k-1} = (n + 1 - 2k)\lambda_0$. Thus we have the result:

The point $C_{k-1} = \frac{G + \gamma_{k-1} P_0}{1 + \gamma_{k-1}}$ which divides the distance $\overline{GP_0}$ at the ratio

(4′) $$\frac{\overline{GC_{k-1}}}{\overline{P_0 C_{k-1}}} = \frac{n + 1 - 2k}{n + 1}$$

is equidistant from the barycentres of all the $k-1$-dimensional simplexes contained in $[P_1 \ldots P_{n+1}]$. The common value of the distance will be found to be determined by

$$4k^2 \overline{C_{k-1} G_{k-1}}^2 = (n + 1 - 2k)^2 \lambda_0 + \lambda_1 + \lambda_2 + \ldots + \lambda_{n+1}.$$

Let us now calculate the distance of the point given by (4′) from the orthocentre O_{k-1} of the $k-1$-dimensional simplex $[P_1 \ldots P_k]$. The barycentric coordinates of O_{k-1} are obviously

(13′) $$\beta_1 = \frac{L}{\lambda_1}; \beta_1 = \frac{L}{\lambda_2}; \ldots; \beta_k = \frac{L}{\lambda_k}; \quad \beta_{k+1} = \ldots = \beta_{n+1} = 0;$$

$$\frac{1}{L} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \ldots + \frac{1}{\lambda_k}.$$

Hence

$$\overline{CO_{k-1}}^2 = \frac{\sum_{i=1}^{k}\left[\alpha\frac{L}{\lambda_i}-\left(1+\frac{\gamma}{\lambda_i}\right)\right]^2\lambda_i+\sum_{j=k+1}^{n+1}\left(1+\frac{\gamma}{\lambda_j}\right)^2\lambda_j}{\alpha^2}=$$

$$=\frac{\sum_{i=1}^{k}\alpha^2\frac{L^2}{\lambda_i}-2k\alpha L-\sum_{i=1}^{k}\frac{2\alpha\gamma L}{\lambda_i}+\sum_{\nu=1}^{n+1}\left(1+\frac{\gamma}{\lambda_\nu}\right)^2\lambda_\nu}{\alpha^2}.$$

With regard to (13′) this expression will be independent of the choice of the vertices $P_1, P_2, \ldots, P_k$, if again $\alpha = 2k$ and $\gamma_{k-1} = (n+1-2k)\lambda_0$, and the common value of the distance $\overline{C_{k-1}O_{k-1}}$ will be found to be equal to $\overline{C_{k-1}G_{k-1}}$. Hence we have the theorem:

The point $C_{k-1} = \dfrac{G+\gamma_{k-1}P_0}{1+\gamma_{k-1}}$, which divides the distance of the barycentre G and the orthocentre P_0 at the ratio $\overline{GC_{k-1}} : \overline{P_0C_{k-1}} = (n+1-2k):(n+1)$, is equidistant from the barycentres and orthocentres of all the $k-1$-dimensional simplexes of $[P_1 \ldots P_{n+1}]$, the common value of the distance being given by

$$4k^2\,\overline{C_{k-1}G_{k-1}}^2 = 4k^2\,\overline{C_{k-1}O_{k-1}}^2 = (n+1-2k)^2\lambda_0+\lambda_1+\lambda_2+\ldots+\lambda_{n+1}$$
$$(k = 1, 2, \ldots, n).$$

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