On the Possibility of Definition by Recursion.

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To Professor Leopold Fejér on his sixtieth birthday February 9, 1940.

In the axiomatic treatment of arithmetic, based on the PEANO axioms[1]), the special arithmetical functions are usually introduced by recursive definitions, the only function occurring in the axioms as a primitive idea being the successor function a'.[2]) For instance, the functions $a+b$, $a.b$, a^b are successively defined by the recursion equations

(1) $$a+0=a, \quad a+b'=(a+b)';$$

(2) $$a.0=0, \quad a.b'=a.b+a;$$

(3) $$a^0=0', \quad a^{b'}=a^b.a.$$

The existence of a function satisfying given recursion equations, which is far from being an immediate consequence of the PEANO axioms[3]), is usually supported by the following heuristic argument.

[1]) G. PEANO, Sul concetto di numero, *Revista di mat.*, 1 (1891), pp. 87—102 and 256—267. The primitive ideas of the PEANO axiomatic system are: "0," "natural number" and "′." The axioms are: (*i*) 0 is a natural number; (*ii*) if a is a natural number, a' is a natural number too; (*iii*) $a'=b'$ implies $a=b$; (*iv*) $a'\neq 0$; (*v*) any hereditary property possessed by 0 is possessed by each natural number. (According to RUSSELL, a property is called hereditary, if, whenever it belongs to a natural number a, it also belongs to a'.) A proof based on (*v*) is called a "proof by induction" (with respect to a).

[2]) After addition has been defined, it can be proved that $a'=a+1$ (1 denoting the natural number $0'$).

[3]) This is shown by the fact that the existence of such a function does not hold in general if axiom (*iii*) or (*iv*) is omitted. Indeed, the finite models

It follows by induction that, for each n, there is a function defined up to n and satisfying up to n the given recursion equations. Indeed, the first of them defines such a function for $n=0$ (thus "up to 0"); and if a function of this kind is defined up to n, the second of the recursion equations allows its definition to be extended, together with its required property, up to n'.

Obviously, this argument is based implicitely on the order relation "$\leq$," for the phrase "up to n" has plainly to be interpreted as "for all $m \leq n$." Now, "$m \leq n$" is usually defined as "there is a natural number k such that $m+k=n$" and "$m+k$" is defined by the recursion equations (1). Hence a vitious circle arises. To avoid it, we have to choose between the following methods.

(*a*) We adjoin the equations (1), together with the primitive idea "+," to the PEANO axioms. This way is, of course, the easiest one, and, if the logic taken as basis of the axiomatic system[4]) is not wide enough to express the idea "there is a function with a given property," it is also the only practicable one[5]). But if we suppose, as we shall do here, that our logic is expressive enough, this method is not satisfactory, for it introduces new axioms which could be avoided.

(a) $0'=1,\ 1'=2,\ 2'=3,\ 3'=1,\ 0\neq 1,\ 0\neq 2,\ 0\neq 3,\ 1\neq 2,\ 1\neq 3,\ 2\neq 3$ and (b) $0'=1,\ 1'=2,\ 2'=0,\ 0\neq 1,\ 1\neq 2$ satisfy the axioms (*i*), (*ii*), (*iv*), (*v*) and (*i*), (*ii*), (*iii*), (*v*) respectively; nevertheless, equations (3) cannot be satisfied in these models, for they would successively imply

$$2^0=1,\ 2^1=1.2=2,\ 2^2=2.2=1,\ 2^3=1.2=2,\ 2^1=2.2=1$$

in (a), and

$$2^0=1,\ 2^1=1.2=2,\ 2^2=2.2=1,\ 2^0=1.2=2$$

in (b), both in contradiction to $1\neq 2$. — On the other hand, the *unicity* of the function satisfying given recursion equations is an immediate consequence of axiom (*v*).

[4]) It is unnecessary to explain that an axiomatic system is not fully determined unless besides the primitive ideas and axioms the logic taken as basis is given. In case of the PEANO axioms, this logic is generally supposed to include the usual properties of the equality. The extent of the logic taken as basis has a great influence on that of the axiomatic theory; e. g. the PEANO axioms are sufficient for the theory of real numbers or for that of natural numbers only, according to the logic taken as basis.

[5]) If we adjoin also the equations (2) to the axioms, further recursive definitions become superfluous, supposing our logic allows us to express the idea "the only natural number of a given property," or, at least, the idea

(*b*) We define "$m \leqq n$" without using functions defined by recursion. This method is due to DEDEKIND[6]); he succeeded, by defining "$m \leqq n$" as "n possesses all hereditary properties possessed by m," in completing the above heuristic argument to an exact proof. However, Dedekind's definition is logically more complicated[7]) and technically less convenient for the proof of the properties of the order relation[8]) than the "usual" definition stated above.

(*c*) We prove the existence theorem in question by another method, without using the order relation. For the particular recursion equations (1) this can be done, as I have shown[9]), by induction with respect to the "parameter" a. This method can be applied to the equations (2) too, but not to arbitrary recursion equations. However, after addition has been introduced, we can define the order relation by the above "usual" definition and then proceed in proving the general existence theorem by Dedekind's method.

This way has the disadvantage of using two entirely different devices to prove the same fact, one in a particular case and then another to settle the general theorem. A method which is free from this disadvantage has been recently given by LORENZEN[10]). He proves the existence theorem at once for arbitrary recursive

"there is a natural number of a given property." Indeed, any recursive definition can be replaced by an explicit one in the first case and, in the second case, by a "contextual" definition giving a meaning to any assertion which contains the function to be defined; see K. GÖDEL, Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I, *Monatshefte für Math. und Phys.*, **38** (1931), pp. 173—198, esp. Satz VII; D. HILBERT und P. BERNAYS, *Grundlagen der Mathematik*, I (Berlin, 1934), pp. 412—421 and 457—460.

[6]) R. DEDEKIND, *Was sind und was sollen die Zahlen*, 5th edition (Braunschweig, 1923), pp. 23—35; *Gesammelte mathematische Werke*, vol. 3 (Braunschweig, 1932), pp. 361—372.

[7]) Indeed, it contains a quantifier ("all") referring to the domain of the *properties of natural numbers*, whereas the "usual" definition contains a quantifier ("there is") referring to the domain of the *natural numbers* only.

[8]) The properties of "$\leqq$" needed in Dedekind's proof are: (1) $n \leqq 0$ implies $n = 0$; (2) $n \leqq m'$ implies $n \leqq m$ or $n = m'$; (3) $n \leqq n$; (4) $n' \leqq n$ does not hold; (5) $0 \leqq n$; (6) $n' \leqq m$ implies $n \leqq m$; (7) $n \leqq m$ implies $n' \leqq m'$.

[9]) See E. LANDAU, Grundlagen der Analysis (Leipzig, 1930), pp. 4—5.

[10]) P. LORENZEN, Die Definition durch vollständige Induktion, *Monatshefte für Math. und Phys.*, **47** (1939), pp. 356—358.

definitions[11]) without using the order relation, but operating with multivalent functions.

In this note, I shall give one more proof, avoiding the order relation like LORENZEN, but also the idea of a multivalent function which is plainly more complicated than that of a univalent function[12]). Instead of it, I use the idea of a function defined only for *some* natural numbers[13]), occurring also in the heuristic argument referred to. Thus we obtain a proof which approaches that argument closer than the former ones and so it seems more natural than any of them.

For simplicity, I shall confine myself to "primitive" recursion equations of the form

$$\begin{aligned} \varphi(0) &= \alpha, \\ \varphi(n') &= \beta(n, \varphi(n)); \end{aligned} \tag{4}$$

here α and β may depend (as already defined functions) also on parameters, in which case the function $\varphi(n)$ to define depends on these parameters too; obviously, (1), (2) and (3) are particular cases of (4). However, the same method can be applied to more general types of arithmetic recursion[14]) as well as to the set-theoretic generalisation of the problem treated by LORENZEN.

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[11]) Moreover, LORENZEN does not restrict the problem to functions of natural numbers with natural numbers as values but treats the problem in a general, set-theoretic form.

[12]) Indeed, a multivalent (arithmetical) function is, as pointed out by LORENZEN, a function of natural numbers whose values are *classes* of natural numbers.

[13]) Instead of saying, a function is not defined for a given argument, we could say, its value is ∞ or any other symbol different from the natural numbers. If we confine ourselves to functions whose values are different from 0, we can use 0 as such a symbol. The general case can be reduced to this case (and so the introduction of a new symbol can be avoided) by replacing the recursion (4) by

$$\begin{aligned} \psi(0) &= \alpha', \\ \psi(n') &= (\beta(n, \delta(\psi(n))))' \end{aligned} \tag{4'}$$

where $\delta(n)$ is defined as 0 for $n=0$ and, for $n \neq 0$, as the natural number whose successor is n (the existence and unicity of this number $\delta(n)$ can be readily proved by axioms (ν) and ($i\nu$)). After having proved the existence of a function ψ satisfying (4'), we set $\varphi(n) = \delta(\psi(n))$ and obtain a function φ satisfying (4).

[14]) See for instance R. PÉTER, Über den Zusammenhang der verschiedenen Begriffe der rekursiven Funktion, *Math. Annalen*, **110** (1934), pp. 612—632;

We call a function ψ defined for some natural numbers a *partial solution of* (4), if

(α) so far as $\psi(0)$ defined, $\psi(0)=\alpha$;

(β) so far as $\psi(n')$ defined, $\psi(n)$ is defined too and $\psi(n')=\beta(n, \psi(n))$.

There are partial solutions of (4), e. g. the function ψ defined nowhere.

We assert

(A) *To each n, there is a partial solution* ψ *of* (4) *for which* $\psi(n)$ *is defined.*

(B) *If* ψ_1 *and* ψ_2 *are partial solutions of* (4), *and if* $\psi_1(n)$ *and* $\psi_2(n)$ *are both defined, then* $\psi_1(n)=\psi_2(n)$.

We prove both assertions by induction. As to (A), the function with

$$\psi(0)=\alpha,$$
$$\psi(n) \text{ undefined for } n\neq 0$$

is a partial solution[15]) with $\psi(0)$ defined. Suppose χ is a partial solution for which $\chi(m)$ is defined; we have to construct a partial solution ω with $\omega(m')$ defined. If $\chi(m')$ is defined, we simply take χ as ω; if not, let[16])

$$\omega(n)=\chi(n) \text{ for } n\neq m',$$
$$\omega(m')=\beta(m, \chi(m)).$$

Obviously ω fulfils (α) because χ did so and[17]) $0\neq m'$. To prove (β) for ω, suppose $\omega(n')$ is defined, i. e. either $n'\neq m'$ and $\chi(n')$ defined, or $n'=m'$. In the first case, $\chi(n)$ is defined too, thus $n\neq m'$ for $\chi(m')$ is undefined; hence $\omega(n)=\chi(n)$ is defined and $\omega(n')=\chi(n')=\beta(n, \chi(n))=\beta(n, \omega(n))$ by (β). In the second case, i. e. if[18]) $n=m$, the same is true, for[19]) $m\neq m'$, thus $\omega(m)=\chi(m)$ is defined and $\omega(m')=\beta(m, \chi(m))=\beta(m, \omega(m))$.

As to (B), so far as $\psi_1(0)$ and $\psi_2(0)$ are both defined, we have $\psi_1(0)=\alpha=\psi_2(0)$ by (α). Suppose (B) holds for n, and

Konstruktion nichtrekursiver Funktionen, *Math. Annalen*, **111** (1935), pp. 42–60; Über die mehrfache Rekursion, *Math. Annalen*, **113** (1936), pp. 489—527.

15) To prove this we have to use $n'\neq 0$, i. e. axiom (*iv*).

16) $\omega(n)=\chi(n)$ means: $\omega(n)$ is undefined if $\chi(n)$ is undefined; and defined to have the value $\chi(n)$ if $\chi(n)$ is defined.

17) Here we use axiom (*iv*) again.

18) Here we use axiom (*iii*).

19) $m\neq m'$ can be easily proved using axioms (*iii*), (*iv*) and (*v*).

$\psi_1(n')$ and $\psi_2(n')$ are both defined; then also $\psi_1(n)$ and $\psi_2(n)$ are defined, thus $\psi_1(n)=\psi_2(n)$ by hypothesis, and

$$\psi_1(n')=\beta(n,\psi_1(n))=\beta(n,\psi_2(n))=\psi_2(n')$$

by (β).[20]

Now let $\varphi(n)$ the common value of the $\psi(n)$ where ψ is any partial solution of (4) defined for n. By (A) and (B), φ is defined everywhere; we prove it satisfies (4).

Indeed, let ψ a partial solution for which $\psi(0)$ is defined; then we have by (α)

$$\varphi(0)=\psi(0)=\alpha.$$

Further, let χ a partial solution with $\chi(n')$ defined; then $\chi(n)$ is defined too; we have $\varphi(n)=\chi(n)$ and

$$\varphi(n')=\chi(n')=\beta(n,\chi(n))=\beta(n,\varphi(n))$$

by (β), which concludes the proof.

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[20]) Note axioms (*iii*) and (*iv*) are not needed to prove (B).