Contribution to Recursive Number Theory.

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To Professor Leopold Fejér, on his sixtieth birthday, with much gratitude.

1. According to a program due te SKOLEM[1]), number theory has to be developed without the use of bound variables (i. e. without the use of expressions such as "for all x . . ." and "there is an x for which . . ."), in basing it on definitions by recursion as well as on proofs by induction. A further restriction would be to confine ourselves to the simplest form of recursion, to the so-called primitive recursion scheme

$$(1) \qquad \begin{cases} \varphi(0) = a \\ \varphi(n') = \alpha(n, \varphi(n)); \end{cases}$$

here $\alpha(n, m)$ is a function defined already, n' denotes the successor of n in the sequence of integers; φ and α may depend on parameters too. The functions obtained from 0 and n' by substitutions and recursions of the form (1) are called *primitively recursive functions.* As I have shown in two previous papers[2]), several methods of definition, more general than (1) and playing a part in number theory[3]), can be reduced to (1).

[1]) Skolem's investigations date from 1919; see TH. SKOLEM, Begründung der elementaren Arithmetik durch die rekurrierende Denkweise, *Videnskapsselskapets Skrifter* (Kristiania), I. Mat.-Naturv. Kl., 1923, No 6, pp. 1—38.

[2]) RÓZSA PÉTER (POLITZER), *a)* Über den Zusammenhang der verschiedenen Begriffe der rekursiven Funktion, *Math. Annalen,* **110** (1934), pp. 612—632; *b)* Über die mehrfache Rekursion, *Math. Annalen,* **113** (1936), pp. 489—527.

[3]) Such methods of definition are (a) the "course-of-values recursion," defining the value of the function for the argument n' by means of several of its preceding values (possibly by means of the whole course of its values

The analogous problem for the *induction*, i.e. that of the reduction of more general proof methods to the simplest scheme of induction:

$$\text{(I)} \qquad \frac{\begin{array}{c} A(0) \\ A(a) \to A(a') \end{array}}{A(a)}$$

(where $A(a)$ stands for a proposition, variable with a, and "$\to$" for "implies"), has been treated in a recent paper of SKOLEM[4]). Here SKOLEM has shown that the schemes occurring in the work of HILBERT and BERNAYS[5]) as formalizations of some methods of proof used in number theory can be reduced to the scheme (I)[6]), using certain functions defined by recursions. These recursions are all of the form (1), except a single recursion of the form

$$\text{(2)} \qquad \begin{cases} \psi(0, b, a) = b \\ \psi(c', b, a) = \varphi(\psi(c, b, a'), a). \end{cases}$$

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for the arguments less than n'), e. g. the famous formula of Euler's for the sum $\sigma(n)$ of the divisors of n:

$$\begin{cases} \sigma(0) = 0 \\ \sigma(n') = \sum\limits_{3i^2+i<2n'} (-1)^{i-1}\sigma\left(n' - \frac{3i^2+i}{2}\right) + \\ \qquad\qquad + \sum\limits_{3i^2-i<2n'} (-1)^{i-1}\sigma\left(n' - \frac{3i^2-i}{2}\right) + \varrho(n'), \end{cases}$$

where $\varrho(n') = (-1)^{i-1}n$ if $n' = \frac{3i^2+i}{2}$ or $n' = \frac{3i^2-i}{2}$ for some i and $\varrho(n') = 0$ otherwise; (b) the "nested recursion," in which the parameters, instead of being kept constant, are substituted by expressions containing functions already defined (e.g. (2) and (3)) or even, in addition, the function to be defined, e.g.

$$\begin{cases} \varphi(0, a) = \alpha(a) \\ \varphi(n', a) = \beta(n, a, \varphi(n, \gamma(n, a, \varphi(n, a)))); \end{cases}$$

(c) the "manifold recursion," i.e. the recursion with respect to more variables simultanously (not to be reduced to primitive recursions in general) in the particular case in which no nesting of expressions, containing the function to be defined, occurs.

4) TH. SKOLEM, Eine Bemerkung über die Induktionsschemata in der rekursiven Zahlentheorie, *Monatshefte für Math. und Phys.*, **48** (1939), pp. 268–276.

5) D. HILBERT und P. BERNAYS, *Grundlagen der Mathematik*, vol. 1. (Berlin, 1934), pp. 343 and 345.

6) This reduction had been carried out in the work of HILBERT and BERNAYS (see the preceding footnote) too, but by means of bound variables.

In the same paper, SKOLEM expressed his doubt of the possibility of such a reduction when confining the recursions to primitive ones.

However, (2) is a particular case of the "nested recursion" which I have proved [loc. cit. [2]), *a*)] not to exceed the class of primitively recursive functions. Yet my proof can not be used directly to dispel Skolem's doubt, for I started by assuming that there is a well determined function satisfying the non-primitive recursion equations in question[7]), e.g. (2) or, to take a more general example,

$$\text{(3)} \qquad \left\{ \begin{aligned} \varphi(n, a) &= \alpha(a) \\ \varphi(n', a) &= \beta(n, a, \varphi(n, \gamma(n, a))) \end{aligned} \right.$$

(α, β, γ primitively recursive functions).

Then I proved that the same function can be obtained from 0 and n' also by substitutions and primitive recursions.

Nevertheless, my proof can be modified, in each case mentioned in the footnote [3]), by first defining a function by combination of primitive recursions and substitutions and then proving that this function satisfies the given more complicated recursion equations (e.g. (2) or (3)). The proofs modified thus do not use any hypothesis going beyond the range of the "primitively recursive number theory," carrying out Skolem's program in the restricted sense; thus, they form a sort of proof-theoretical strengthening of the original proofs.

2. I shall develop such a strengthened proof in the case of scheme (3). For this scheme, I only remarked[8]) that it can be reduced to (1) in a much easier method than the general nested recursion scheme; but since (3) is a particular case of the latter, I did not give this easier method in details. Now, I shall develop it in its strengthened form in the above sense.

I shall need the function $a \dot{-} n$, denoting $a - n$ for $a \geqq n$ and

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[7]) The usual existence proofs of functions satisfying given (primitive or non-primitive) recursion equations require bound (function) variables, see e.g. L. KALMÁR, On the Possibility of Definition by Recursion, *these Acta*, **9** (1939), pp. 227—232. According to Skolem's program, we may take primitive recursion equations as axioms; but to prove the existence of a function satisfying given non-primitive recursion equations, the only way is to construct an expression by means of substitutions, starting from functions defined by primitive recursion equations, and to prove that it satisfies them.

[8]) See loc. cit. [2]) *a*), p. 626, footnote.

0 for $a < n$, [see loc. cit. [2]), *a*)] shown to be primitively recursive by the definitions

$$\begin{cases} \delta(0) = 0 \\ \delta(n') = n \end{cases}$$

and

$$\begin{cases} a \dot{-} 0 = a \\ a \dot{-} n' = \delta(a - n). \end{cases}$$

I shall refer, in footnotes, to the properties of this function and those of the relation $a \leqq b$ (defined as $a \dot{-} b = 0$) which are needed in the proof; they can be readily proved by means of scheme (I).

Using the function γ occuring in (3), I define a function $\mu(m, n, a)$ by means of the recursion

$$\begin{cases} \mu(0, n, a) = a \\ \mu(m', n, a) = \gamma(n \dot{-} m', \mu(m, n, a)) \end{cases} \tag{4}$$

of the form (1). I prove by means of scheme (I) that

$$i \leqq n \rightarrow \mu(i', n', a) = \mu(i, n, \gamma(n, a)). \tag{5}$$

First, this holds if $i = 0$, for we have[9]) by (4)

$$\mu(0', n', a) = \gamma(n' \dot{-} 0', \mu(0, n', a)) = \gamma(n, a) = \mu(0, n, \gamma(n, a)).$$

Further, I prove that

$$(i \leqq n \rightarrow \mu(i', n', a) = \mu(i, n, \gamma(n, a))) \rightarrow$$
$$\rightarrow (i' \leqq n \rightarrow \mu(i'', n', a) = \mu(i', n, \gamma(n, a)))$$

which can be readily transformed into

$$\begin{aligned} &(i' \leqq n \& (i \leqq n \rightarrow \mu(i', n', a) = \mu(i, n, \gamma(n, a)))) \rightarrow \\ &\qquad \rightarrow \mu(i'', n', a) = \mu(i', n, \gamma(n, a)) \end{aligned} \tag{6}$$

(of course, "&" means "and").

By (4) we have[9])

$$\mu(i'', n', a) = \gamma(n' \dot{-} i'', \mu(i', n', a)) = \gamma(n \dot{-} i', \mu(i', n', a)),$$

whence[10])

$$\begin{aligned} &(i' \leqq n \& (i \leqq n \rightarrow \mu(i', n', a) = \mu(i, n, \gamma(n, a)))) \rightarrow \\ &\qquad \rightarrow \mu(i'', n', a) = \gamma(n \dot{-} i', \mu(i, n, \gamma(n, a))); \end{aligned} \tag{7}$$

as, by (4),

$$\gamma(n \dot{-} i', \mu(i, n, \gamma(n, a))) = \mu(i', n, \gamma(n, a)),$$

(7) proves (6) and, by scheme (I), also (5).

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[9]) $n' \dot{-} k' = n \dot{-} k$.

[10]) $k' \leqq n \rightarrow k \leqq n$.

Now, using the functions α and β occurring in (3), I define a function $\psi(m, n, a)$ by the recursion

$$(8) \quad \begin{cases} \psi(0, n, a) = \alpha(\mu(n, n, a)) \\ \psi(m', n, a) = \beta(m, \mu(n \dot{-} m', n, a), \psi(m, n, a)), \end{cases}$$

which is also an instance of (1). I prove that

$$(9) \qquad m \leqq n \rightarrow \psi(m, n', a) = \psi(m, n, \gamma(n, a)).$$

First, this holds for $m = 0$; indeed, by (8),

$$\psi(0, n', a) = \alpha(\mu(n', n', a)),$$

whereas (5) gives[11] for $i = n$

$$\mu(n', n', a) = \mu(n, n, \gamma(n, a))$$

which leads, using (8), to

$$\psi(0, n', a) = \alpha(\mu(n, n, \gamma(n, a))) = \psi(0, n, \gamma(n, a)).$$

Further, I prove that

$$(m \leqq n \rightarrow \psi(m, n', a) = \psi(m, n, \gamma(n, a))) \rightarrow \\ \rightarrow (m' \leqq n \rightarrow \psi(m', n', a) = \psi(m', n, \gamma(n, a))).$$

The latter relation can be readily transformed into

$$(10) \quad (m' \leqq n \& (m \leqq n \rightarrow \psi(m, n', a) = \psi(m, n, \gamma(n, a)))) \rightarrow \\ \rightarrow \psi(m', n', a) = \psi(m', n, \gamma(n, a)).$$

Now, (8) gives[9])

$$\psi(m', n', a) = \beta(m, \mu(n' \dot{-} m', n', a), \psi(m, n', a)) = \\ = \beta(m, \mu(n \dot{-} m, n', a), \psi(m, n', a));$$

on the other hand, (5) gives[12][13] for $m' \leqq n$ by the substitution $i = n \dot{-} m'$

$$\mu(n \dot{-} m, n', a) = \mu((n \dot{-} m')', n', a) = \mu(n \dot{-} m', n, \gamma(n, a))$$

which proves

$$\psi(m', n', a) = \beta(m, \mu(n \dot{-} m', n, \gamma(n, a)), \psi(m, n', a))$$

for $m' \leqq n$. Thus we have[10]

$$(11) \quad (m' \leqq n \& (m \leqq n \rightarrow \psi(m, n', a) = \psi(m, n, \gamma(n, a)))) \rightarrow \\ \rightarrow \psi(m', n', a) = \beta(m, \mu(n \dot{-} m', n, \gamma(n, a)), \psi(m, n, \gamma(n, a)));$$

on the other hand, by (8),

$$\beta(m, \mu(n \dot{-} m', n, \gamma(n, a)), \psi(m, n, \gamma(n, a))) = \psi(m', n, \gamma(n, a)),$$

therefore, (11) proves (10) and, by scheme (I), also (9).

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[11]) $n \leqq n$.

[12]) $n \dot{-} k \leqq n$.

[13]) $m' \leqq n \rightarrow n \dot{-} m = (n \dot{-} m')'$.

(9) gives for[11] $m = n$

(12) $$\psi(n, n', a) = \psi(n, n, \gamma(n, a)).$$

Finally, I assert that the function

(13) $$\varphi(n, a) = \psi(n, n, a)$$

satisfies the equations (3).

Indeed, we have by (13), (8) and (4)

$$\varphi(0, a) = \psi(0, 0, a) = \alpha(\mu(0, 0, a)) = \alpha(a);$$

further by (13) and (8),

$$\varphi(n', a) = \psi(n', n', a) = \beta(n, \mu(n' \dot{-} n', n', a), \psi(n, n', a))$$

and here we have[14] by (4)

$$\mu(n' \dot{-} n', n', a) = \mu(0, n', a) = a,$$

thus, using (12) and (13),

$$\varphi(n', a) = \beta(n, a, \psi(n, n, \gamma(n, a))) = \beta(n, a, \varphi(n, \gamma(n, a))),$$

which proves our assertion. Obviously $\mu(m, n, a)$ and $\psi(m, n, a)$ are primitively recursive functions and hence $\varphi(n, a)$, obtained from $\psi(m, n, a)$ by substitution, is also a primitively recursive function, as stated above.

3. The function defined by (2) was used by SKOLEM in the proof of the reducibility of the scheme

(II) $$\frac{\begin{array}{l} A(b, 0) \\ A(\varphi(b, a), a) \to A(b, a') \end{array}}{A(b, a)}$$

to scheme (I). Definition (2) being an instance of scheme (3), Skolem's proof can be carried out also in the case of confining the recursions to primitive ones.

This fact can also be shown directly, without the roundabout way through functions defined by non-primitive recursions, by a slight modification of Skolem's proof; see my review on Skolem's paper (loc. cit. [4])), forthcoming in *The Journal of Symbolic Logic.*

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[14]) $k \dot{-} k = 0.$