On Upper Semi-Continuous Collections.

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Introduction.

During the course of the years 1924 to 1927, R. L. MOORE, P. ALEXANDROFF and B. DE KERÉKJÁRTÓ appear to have independently introduced the notion of an upper semi-continuous collection[1]). Apart from the fundamental ideas, however, their work does not overlap: the space, the terminology and the application is different in each case. Nevertheless, on account of the obvious importance of the concepts introduced by them, it is of interest to observe that their definitions, though worded in entirely different fashions, become equivalent if applied in a certain conveniently chosen abstract space.

This equivalence is rather obvious, but the purpose of this note is to establish it in a way which is not suggested immediately by their work. Indeed, we shall show that the definitions proposed represent in a sense the unique answer to a certain simple and fundamental question, and hence are necessarily equivalent. In other words our purpose is more than to show the identical character of the definitions, we shall show that for *a priori* reasons there is a sort of compulsion about the matter.

For convenience and clarity we shall start with the definitions to be used.

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1) R. L. MOORE, Concerning Upper Semi-Continuous Collections of Continua which do not Separate a Given Continuum, *Proceedings National Academy of Sciences,* **10** (1924), pp. 356—360; PAUL ALEXANDROFF, Über stetige Abbildungen kompakter Räume, *Math. Annalen,* **96** (1926), pp. 555—571; B. DE KERÉKJÁRTÓ, Involutions et surfaces continues, *these Acta,* **3** (1927), pp. 49—67. Cf. also B. DE KERÉKJÁRTÓ, On Parametric Representations of Continuous Surfaces, *Proceedings National Academy of Sciences,* **10** (1924), pp. 267–271.

I. Preliminary notions.

1.1. *Partitions.* Let S be any set and suppose the elements are denoted generically by $a, b, c, \ldots$ etc.

By a *partition* $\mathfrak{P}$ of S we mean a collection of mutually exclusive subsets α, of S, filling up S. In other words, no element is in two sets of the collection $\mathfrak{P}$, but every element is in some set of $\mathfrak{P}$. A subset of S which is in the collection $\mathfrak{P}$ is known as a *compartment* of $\mathfrak{P}$.

1.2. *Equivalence.* Every partition gives rise to the following binary relationship: $a \sim b(\mathfrak{P})$ meaning that a and b are elements of the same compartment of $\mathfrak{P}$. This binary relationship will be called an *equivalence* $E(\mathfrak{P})$ since it satisfies the familiar postulates for an equivalence: $(E_1) a \sim b(\mathfrak{P})$ or a non-$\sim b(\mathfrak{P})$. $(E_2) a \sim a(\mathfrak{P})$. $(E_3) a \sim b(\mathfrak{P})$ implies $b \sim a(\mathfrak{P})$. $(E_4) a \sim b(\mathfrak{P})$ and $b \sim c(\mathfrak{P})$ imply $a \sim c(\mathfrak{P})$.

Conversely, if we are given an equivalence there is a unique partition $\mathfrak{P}$ of S which generates it.

1.3. *The set Σ.* Every partition $\mathfrak{P}$ also gives rise to a set Σ whose elements are the compartments of $\mathfrak{P}$. Each compartment plays a dual rôle. It is a subset of S and is also an element of Σ. We shall use the notation α when we consider a compartment as a subset of S and reserve the notation $[\alpha]$ to denote that the same compartment is considered as an element of Σ.

1.4. *The transformation T.* Finally every partition gives rise to a transformation from S to Σ, a transformation which associates with an element a of S that element $[\alpha]$ of Σ which as a set α contains a. Symbolically,

$$[\alpha] = T(a) \text{ means } a \in \alpha.$$

In terms of the transformation T we see that $a \sim b(\mathfrak{P})$ means $T(a) = T(b)$.

1.5. *L^* spaces*[2]. An L^* space is a limit space (i. e., a space in which convergent sequences and their limits are defined) satisfying the following conditions

[2]) Conditions L_1 and L_2 are those of Fréchet, Sur quelques points du Calcul fonctionnel, *Rendiconti Circolo Mat. di Palermo*, **22** (1906), pp. 1—74. Condition L_3 is from P. Alexandroff and P. Urysohn, Une condition nécessaire et suffisante pour qu'une classe (L) soit une classe (D), *Comptes rendus Paris*, **177** (1923), pp. 1274—1276. See also C. Kuratowski, *Topologie I*, (Warszawa, 1933), chapter II.

L_1. $a, a, a, \ldots \to a$.

L_2. If $a_n \to a$ then any subsequence $a_{n_j} \to a$.

L_3. If $a_n \not\to a$, then there is a subsequence $\{a_{n_j}\}$ such that no subsequence of this subsequence converges to a. (The symbol $\not\to$ means that a_n does not converge to a.)

We shall ultimately be interested only in *compact* L^* spaces, that is in L^* spaces having the property that *any sequence has a convergent subsequence.* In other words a compact L^* space is a limit space satisfying L_1, L_2, and

L_3'. If $a_n \not\to a$, then there is a subsequence $a_{n_j} \to b \neq a$.

Given a set S by the topologization of S we shall mean the assigning of convergent sequences and limits in S so that the resulting limit space is a compact L^* space.

1. 6. *Compatibility.* Let us return for a moment to the equivalence mentioned in 1. 2. If $\mathfrak{P}$ is a partition of a compact L^* space S, we shall say that the equivalence $E(\mathfrak{P})$ is *compatible with the topology of* S if and only if $a_n \to a$, $b_n \to b$, $a_n \sim b_n(\mathfrak{P})$ for $n = 1, 2, \ldots$, imply $a \sim b(\mathfrak{P})$.

II. The theorems.

Referring to the work of MOORE, ALEXANDROFF and KERÉKJÁRTÓ mentioned in the introduction, the ideas which seem to be common to all three are the space S, the partition $\mathfrak{P}$, and the set Σ which each topologizes in his own way, and later verifies that the transformation T is continuous if and only if $\mathfrak{P}$ is properly restricted.

Looking at the common features of their work from an abstract point of view, the real issue at the center of things appears to suggest itself in the form of the following question:

Given S_1, a compact L^* space, and S_2, an untopologized set of elements, and a single valued transformation $a_2 = T(a_1)$ from S_1 to S_2, in how many ways can S_2 be topologized so that T is continuous?

Upon answering this question things become excessively simple, and, pleasingly enough, the answer is both direct and complete: the topologization is unique[3]).

[3]) Instead of the direct proof we shall give, we might have used a result of F. HAUSDORF, *Mengenlehre,* (3rd edition, Berlin, 1935), p. 194. If S_1 and S_2 are metric spaces then T is continuous if and only if the complete model in S_1 of a closed set in S_2 is closed. The remarks also apply to *compact* L^* spaces and in addition the image of a closed set is closed, so that closed sets and therefore limits are determined *a priori* in S_2 by T and the topology of S_1.

First of all, an immediate example shows that if S_1 is merely required to be an L^* space, then the topologization of S_2 compatible with continuity on the part of T need not be unique. On the x-axis, let S_1 be the unit interval $0 < x \leqq 1$ and let the topologization of S_1 be taken from the x-axis itself. That is, the usual meaning of convergence and limit holds. Let S_2 be the same set of elements and let $a_2 = T(a_1)$ mean $a_2 = a_1$. We assert that S_2 can be topologized in a non-denumerable number of ways, and in each instance T is continuous. Let k be a fixed number $0 < k \leqq 1$. Take the usual meaning of convergence and limit for all sequences of elements of S_2 except for those which would, when considered as sequences of points on the x-axis, converge to the limit 0. To each such sequence assign the limit k. It is clear that S_2 is now a compact L^* space and T is continuous.

2. 1. *Uniqueness of topologization.* The above example indicates the importance of compactness on the part of S_1 if one is to hope for uniqueness. Stated in terms of S and Σ the general statement is as follows.

Theorem. *If S is a compact L^* space, and Σ can be topologized at all so that T is continuous, then the topologization of Σ is univocally determined, namely, $[\alpha_n] \rightarrow [\alpha]$ implies $\overline{\lim}\, \alpha_n \subset \alpha$; and conversely, $\overline{\lim}\, \alpha_n \subset \alpha$ implies $[\alpha_n] \rightarrow [\alpha]$.*[4])

Proof. If $a_n \in \alpha_n$ for $n = 1, 2, \ldots$ and $a_{n_j} \rightarrow a$, then $T(a_{n_j}) \rightarrow T(a)$ since T is continuous. But $T(a_{n_j}) = [\alpha_{n_j}] \rightarrow [\alpha]$ by L_2 on Σ. Therefore $[\alpha] = T(a)$ since the limit is unique, and so $a \in \alpha$.

Conversely, if $\overline{\lim}\, \alpha_n \subset \alpha$ but $[\alpha_n] \not\rightarrow [\alpha]$ then it is no restriction to suppose, using L_3 on Σ, that no subsequence of $\{[\alpha_n]\}$ converges to $[\alpha]$. But $\overline{\lim}\, \alpha_n \subset \alpha$ implies, since S is compact, that there exists $a_n \in \alpha_n$ for $n = 1, 2, \ldots$ such that a subsequence $a_{n_j} \rightarrow a \in \alpha$. Since T is continuous, $T(a_{n_j}) \rightarrow T(a)$, which means $[\alpha_{n_j}] \rightarrow [\alpha]$.

The reader will have observed that the sets of the partition $\mathfrak{P}$ are entirely unrestricted, and that the statement of the theorem might have been made much stronger since we have used only

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[4]) *Cf.* C. Kuratowski, Sur les décompositions semi-continues d'espaces métriques compacts, *Fundamenta Math.*, 11 (1928), pp. 169—185, especially p. 171, where the topologization is stated in exactly this form. However, Kuratowski does not state the fact that this is the *only* way to topologize. Remark: $a \in \overline{\lim}\, \alpha_n$ if and only if there exists $a_n \in \alpha_n$ for $n = 1, 2, \ldots$, and $a_{n_j} \rightarrow a$.

compactness on the part of S, and L_2 and L_3 on Σ, but have nowhere assumed that L_3 holds (which means we do not need compactness on Σ).

2. 2. *Existence of topologization.* The theorem of the preceding section is a uniqueness theorem. It points out that if the topologization is possible then the method must be precisely as given above. The question which naturally follows is *under what conditions is the topologization possible?*

MOORE, ALEXANDROFF and KERÉKJÁRTÓ each has his own condition which guarantees the possibility of topologization compatible with the continuity of T. Since the conditions are necessary and sufficient they are necessarily equivalent, and 1. 2 shows that the same is true of the different methods of topologization.

In the terminology of this note we have at our disposal an immediate necessary and sufficient condition which may be stated as follows.

Theorem. *Σ can be topologized so that T is continuous if and only if the equivalence $E(\mathfrak{P})$ is compatible* (see 1. 6) *with the topology of S.*

Proof. Take $a_n \to a$ and $b_n \to b$, such that $a_n \sim b_n(\mathfrak{P})$; i. e., $T(a_n) = T(b_n)$ for $n = 1, 2, \ldots$. Since the transformation is continuous $T(a_n) \to T(a)$, and $T(b_n) \to T(b)$. Hence $T(a) = T(b)$ which means that $a \sim b(\mathfrak{P})$. In other words $E(\mathfrak{P})$ is compatible with the topology of S.

This proves the necessity of the condition. The sufficiency follows directly and is left to the reader.

1. 3 *Upper semi-continuity.* In concluding this note we wish to state the definition of upper semi-continuous collection essentially as introduced by MOORE. A partition $\mathfrak{P}$ is said to be upper semi-continuous if $a_n \to a \in \alpha$, $a_n, b_n \in \alpha_n$ for $n = 1, 2, \ldots$ and $b_{n_j} \to b$ imply that $b \in \alpha$. As a result of the above remarks this is equivalent, of course, to the statement that $E(\mathfrak{P})$ is compatible with the topology of S. The direct proof of this statement is left to the reader.

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