On a theorem of S. Bernstein.

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Let $f(x)$ be continuous in the interval $-1 \leqq x \leqq +1$ and let us denote

(1) $$L_n[f; x] \qquad n = 1, 2, \ldots$$

the n-th Lagrange interpolation polynomial of the function $f(x)$ corresponding to the Chebysheff abscissas. Then (1) is the unique polynomial of degree $n-1$ which assumes the values $f(x_1^{(n)})$, $f(x_2^{(n)}), \ldots, f(x_n^{(n)})$ at the zeros

(2) $$x_1^{(n)} = \cos\frac{\pi}{2n},\ x_2^{(n)} = \cos 3\frac{\pi}{2n}, \ldots, x_n^{(n)} = \cos(2n-1)\frac{\pi}{2n}$$

of the n-th Chebysheff polynomial $T_n(x) = \cos(n \arccos x)$. It is known that the sequence (1) is not convergent for every continuous function[1]); moreover there exist continuous functions for which the sequence (1) is divergent everywhere[2]) in the interval $-1 \leqq x \leqq +1$. S. BERNSTEIN proved[3]) that we can modify (1) by choosing suitably λn of the abscissas (2) and changing there suitably the values of f, so that the new sequence of interpolation polynomials be

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1) G. FABER, Über die interpolatorische Darstellung stetiger Funktionen, *Jahresbericht der Deutschen Math. Vereinigung*, **23** (1914), p. 190–210; S. BERNSTEIN, Sur la limitation des valeurs ..., *Bulletin Academy of Sciences de l'URSS*, 1931, p. 1025–1050.

2) G. GRÜNWALD, Über Divergenzerscheinungen der Lagrangeschen Interpolationspolynome stetiger Funktionen, *Annals of Math.*, **37** (1936), p. 908–918. See also J. MARCINKIEWICZ, Sur la divergence des polynomes d'interpolation, *these Acta*, **7** (1937), p. 131–135.

3) S. BERNSTEIN, Sur une modification de la formule d'interpolation de Lagrange, *Communications de la Société Math. de Kharkow*, (4) **5** (1932), p. 49–57.

uniformly convergent in the interval $-1 \leqq x \leqq +1$ for every continuous function, where $\lambda > 0$ is an arbitrary small but fixed number.

In this note, we prove that the theorem of BERNSTEIN is exact in the sense that, instead of fixed λ, we cannot take a sequence $\lambda = \lambda_n$ for which $\lambda_n \to 0$ if $n \to \infty$. In other words, if $\lambda_n \to 0$ for $n \to \infty$, if we choose $n\lambda_n$ abscissas from (2) and we change the values of f assumed here arbitrarily, then there exist continuous functions for which the modified sequence of interpolation polynomials is not uniformly convergent in the interval $-1 \leqq x \leqq +1$.

The following explicit form of (1) is known:

$$(3)\quad L_n[f; x] = \sum_{k=1}^{n} (-1)^{k+1} \frac{T_n(x)\sqrt{1-(x_k^{(n)})^2}}{n(x-x_k^{(n)})} f(x_k^{(n)}) = \sum_{k=1}^{n} l_k(x) f(x_k^{(n)}).$$

The modified interpolation polynomial is

$$(4)\qquad B_n[f; x] = \sum_{k=1}^{n} A_k^{(n)} l_k(x),$$

where

$$(5)\qquad A_{i_1}^{(n)}, A_{i_2}^{(n)}, \ldots, A_{i_\mu}^{(n)} \qquad \mu = n\lambda_n$$

are arbitrary numbers and

$$(6)\qquad A_{i_{\mu+1}}^{(n)} = f(x_{i_{\mu+1}}^{(n)}),\ A_{i_{\mu+2}}^{(n)} = f(x_{i_{\mu+2}}^{(n)}), \ldots, A_{i_n}^{(n)} = f(x_{i_n}^{(n)})$$

($i_1, i_2, \ldots, i_n$ being an arrangement of the numbers $1, 2, \ldots, n$).

Evidently we can suppose that

$$(7)\qquad \overline{\lim}\, |A_k^{(n)}| \leqq 1, \qquad k = 1, 2, \ldots, \mu = \lambda_n n;\ n = 1, 2, \ldots$$

Let us divide the interval $-1 \leqq x \leqq +1$ into $\mu + 1$ equal parts. There is at least one part not containing any of the abscissas $x_{i_1}^{(n)}, x_{i_2}^{(n)}, \ldots, x_{i_\mu}^{(n)}$. Let $x_j^{(n)}, x_{j+1}^{(n)}$ be the two Tschebyscheff abscissas intercepting the centre of this interval and put $x_0 = \cos j\frac{\pi}{n}$. Then we have[4])

$$(8)\qquad \sum_{k=1}^{\mu} |l_{i_k}(x_0)| = \frac{1}{n}\sum_{k=1}^{\mu} \frac{\sqrt{1-(x_{i_k}^{(n)})^2}}{|x_0 - x_{i_k}^{(n)}|} <$$

$$< \frac{1}{n}\sum_{k=1}^{\mu} \frac{1}{|x_0 - x_{i_k}|} < \frac{c}{n}\sum_{k=1}^{\mu} \frac{n}{\mu + k} < c.$$

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[4]) $c > 0$ denotes an absolute constant, not necessarily the same one each time it occurs.

Let $f_n(x)$ be a continuous function in the interval $-1 \leqq x \leqq +1$ for which

(9) $$f_n(x_k^{(n)}) = \operatorname{sign} l_k(x_0), \qquad k = 1, 2, \ldots, n.$$

From (4), (5), (6), (7), (8), (9) it follows that

$$|B_n[f_n; x_0]| = \left| \sum_{k=1}^{n} A_k^{(n)} l_k(x_0) \right| =$$

(10) $$= \left| \sum_{k=1}^{n} f_n(x_k^{(n)}) l_k(x_0) + \sum_{k=1}^{\mu} (A_{i_k}^{(n)} - f(x_{i_k}^{(n)})) l_{i_k}(x_0) \right| \leqq$$

$$\leqq \sum_{k=1}^{n} |l_k(x_0)| - 2 \sum_{k=1}^{\mu} |l_{i_k}(x_0)| < \sum_{k=1}^{n} |l_k(x_0)| - c.$$

Since[5])

(11) $$\sum_{k=1}^{n} |l_k(x)| > \frac{1}{\pi} \log n \, |T_n(x)|$$

and so

(12) $$\sum_{k=1}^{n} |l_k(x_0)| > \frac{1}{\pi} \log n,$$

we have

(13) $$|B_n[f_n; x_0]| > c \log n.$$

With the aid of (13) it is easy to prove that the sequence of interpolation polynomials $B_n[f; x]$ is not uniformly convergent if $f(x)$ is the following continuous function

(14) $$f(x) = \sum_{\nu=1}^{\infty} \frac{f_{n_\nu}(x)}{\nu^2}$$

and if the sequence n_ν is suitably chosen.

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[5]) G. GRÜNWALD, Über Divergenzerscheinungen der Lagrangeschen Interpolationspolynome, *these Acta*, 7 (1935), p. 206—221.