A remark on the length of the circle and on the exponential function.

By E. EGERVÁRY in Budapest.

The length of a curve is generally defined as the limit of the length of an inscribed polygon whose sides tend to zero. To justify this definition, it must be shown, that this limit is independent of the mode of subdivision of the curve.

In elementary geometry, as a rule, the length of the circ'e is defined as the limit of a particular sequence of regular inscribed polygons, and one proves afterwards, indirectly[1]) (if at all), that any other sequence tends to the same limit.

In the present note I intend to give a direct and elementary proof of the

Theorem I. *The length of the inscribed polygon in a circle tends to a limit, when the number of sides increases indefinitely in such a way that all the sides tend to zero.*

As any two such sequences of inscribed polygons may be united to a third sequence of the same kind, Theorem I proves at once that the limit does not depend on the choice of the sequence of polygons.

Theorem I is based upon the following

Lemma I[2]). *If A and B are two inscribed polygons which contain the center of the circle in their interior and are such that the largest side of B is smaller than the smallest side of A, then A is shorter than B.*

---

1) To prove the existence of the length of a circle, most of the writers see themselves compelled to consider simultaneously inscribed *and* circumscribed polygons. See e. g. G. H. HARDY, *Pure Mathematics* (1908), p. 276; J. HADAMARD, *Géométrie élémentaire* (1925), pp. 170–172; H. WEBER—J. WELLSTEIN, *Encykl. d. Elementarmath.* (1907), II, pp. 269–273; KÜRSCHÁK J., *Analizis* (1923), pp. 122—128.

2) Lemma I (proposed by the author as a problem in the *Math. és Fiz. Lapok,* **50** (1943), pp. 377—380) is equivalent to a theorem of M. DEHN, *Norske Vid. Selsk. Forh.,* **13** (1941), pp. 103–106.

The well known theorem, that the length of a regular polygon increases steadily with the number of sides, is an obvious corollary of Lemma I.

The method may be extended without any difficulty to the definition of the length of a circular arc.

The number e is generally defined as $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. Here I will suggest a more general definition, which relates to the former one in the same fashion as the definition of the length of the circle by means of general sequences to that by means of regular polygons. Indeed, the number e may be defined in the following way:

Theorem II. *If $\alpha_1, \alpha_2, \dots, \alpha_n$ are positive numbers, whose sum is equal to* 1, *then $\prod_{\nu=1}^{n}(1+\alpha_\nu)$ tends to a limit, when n increases in such a way that all of the numbers α_ν tend to zero.*

As any two such sequences of n-tuples $(\alpha_1, \alpha_2, \dots, \alpha_n)$ may be united to a third sequence of the same kind, Theorem II proves at once that the limit does not depend on the choice of the sequence. By taking $\alpha_1=\alpha_2=\dots=\alpha_n=\frac{1}{n}$, it is obvious that *the limit in question is equal to e*, which is now defined in the same general way, as π by means of general sequences of polygons.

Theorem II is based upon the following

Lemma II. *If $\alpha_1, \alpha_2, \dots, \alpha_m$ and $\beta_1, \beta_2, \dots, \beta_n$ are two sets of positive numbers such that $\sum_1^m \alpha_\mu = \sum_1^n \beta_\nu$ and* $\max\beta_\nu < \min\alpha_\mu$, *then*

$$\prod_1^n (1+\beta_\nu) > \prod_1^m (1+\alpha_\mu).$$

The well known theorem, that $\left(1+\frac{1}{n}\right)^n$ increases steadily with n is an obvious consequence of Lemma II.

Our definition of e may be extended to e^x (by replacing the assumption $\Sigma\alpha_\nu = 1$ by $\Sigma\alpha_\nu = x$) and has the advantage, that the fundamental identity $e^{x+y} = e^x e^y$ is an obvious consequence of it.

Lemma I may be proved by means of the following inequality:[3])

If the points X and Y of the

Lemma II may be proved by means of the following inequality:

If the positive numbers x_1, x_2,

[3]) See also J. Kürschák, Über dem Kreise ein- und umgeschriebene Vielecke, *Math. Annalen,* **30** (1887), pp. 578–581.

circular arc $\widehat{AB}$ are such that

$$\left.\begin{aligned}\widehat{AX}>\widehat{AY}>\widehat{XB},\\ \widehat{AX}>\widehat{YB}>\widehat{XB},\end{aligned}\right\} \qquad (1_{\text{I}})$$

then

$$\overline{AY}+\overline{YB}>\overline{AX}+\overline{XB}. \qquad (2_{\text{I}})$$

Indeed, if $\overline{AX}$ and $\overline{BY}$ meet in Z, then, owing to the similarity of $\triangle AYZ$ and $\triangle BXZ$,

$$\overline{BX}=q\overline{AY}$$
$$\overline{XZ}=q\overline{YZ}$$
$$\overline{BZ}=q\overline{AZ}$$

where $q<1$. Consequently

$$\overline{AY}+\overline{YB}-(\overline{AX}+\overline{XB})=$$
$$=(\overline{AY}+\overline{YZ}-\overline{AZ})(1-q)>0,$$

qu. e. d.

Let us now denote the cyclic succession of the vertices A_μ, B_ν of the inscribed polygons[4]) by

$$A_0=B_0\prec B_1\prec\ldots\prec$$
$$\prec B_{\nu_1}\prec A_1\leqq B_{\nu_1+1}\prec\ldots\prec A_0=B_0.$$

Then by repeated application of the inequality (2_{I}) and of the triangular inequality (and using the abbrevation $\overline{XY}+\overline{YZ}+\ldots+$ $+\overline{WT}=\overline{XY\ldots T}$)

$$\overline{A_0B_1\ldots B_{\nu_1}B_{\nu_1+1}\ldots B_0}\geqq$$
$$\geqq\overline{A_0B_{\nu_1}B_{\nu_1+1}\ldots B_0}>$$
$$>\overline{A_0A_1B_{\nu_1+1}\ldots B_0}\geqq$$
$$\geqq\overline{A_0A_1B_{\nu_2}\ldots B_0}>$$
$$>\ldots>\overline{A_0A_1A_2\ldots A_0}.$$

We have thus proved the Lemma I.

Let us now consider a sequence of inscribed polygons P_n of length l_n and such that the largest side of P_n tends to zero as $n\to\infty$.

y_1, y_2 are such that

$$\left.\begin{aligned}x_1+x_2=y_1+y_2,\\ x_1>y_1>x_2,\\ x_1>y_2>x_2,\end{aligned}\right\} \qquad (1_{\text{II}})$$

then

$$y_1y_2>x_1x_2. \qquad (2_{\text{II}})$$

Indeed, by (1_{II}) we have

$$y_1y_2-x_1x_2=$$
$$=y_1y_2-(y_1+y_2-x_2)\,x_2=$$
$$=(y_1-x_2)\,(y_2-x_2)>0,$$

qu. e. d.

Let us now denote the succession of the sums

$$A_\mu=\alpha_1+\alpha_2+\ldots+\alpha_\mu,$$
$$B_\nu=\beta_1+\beta_2+\ldots+\beta_\nu$$

in ascending order by

$$0<B_1<B_2<\ldots<B_{\nu_1}<A_1\leqq$$
$$\leqq B_{\nu_1+1}<\ldots<A_m=B_n.$$

Then by repeated application of the obvious inequality

$$(1+x_1)(1+x_2)>1+x_1+x_2$$
$$(x_1,\ x_2>0)$$

and of (2_{II}) we get

$$(1+\beta_1)\ldots(1+\beta_{\nu_1})(1+\beta_{\nu_1+1})\ldots(1+\beta_n)\geqq$$
$$\geqq(1+B_{\nu_1})(1+\beta_{\nu_1+1})\ldots(1+\beta_n)>$$
$$>(1+\alpha_1)(1+B_{\nu_1+1}-\alpha_1)\ldots(1+\beta_n)\geqq$$
$$>\ldots>(1+\alpha_1)(1+\alpha_2)\ldots(1+\alpha_n).$$

We have thus proved the Lemma II.

Let us now consider a sequence of products $l_n=\prod_{\nu=1}^{n}(1+\alpha_{n\nu})$ such that

$$\alpha_{n\nu}>0,\ \sum_{\nu=1}^{n}\alpha_{n\nu}=1$$

and the largest factor of l_n tends to 1, as $n\to\infty$.

4) We suppose, without loss of generality, that $A_0=B_0$.

If the sequence of the numbers l_n (which is in both cases bounded[5])) would be divergent, then one could select two convergent partial sequences

$$l_{\mu_1}, l_{\mu_2}, \ldots, l_{\mu_k},$$
$$l_{\nu_1}, l_{\nu_2}, \ldots, l_{\nu_k},$$

with distinct limits. But h being given *arbitrarily*, we can find a number k such that

the largest side of P_{ν_k} is smaller than the smallest one of P_{μ_h}, and, owing to the Lemma I,	the largest factor of l_{ν_k} is smaller than the smallest one of l_{μ_h}, and, owing to the Lemma II,

$$l_{\nu_k} > l_{\mu_h}.$$

Consequently

$$\lim_{k\to\infty} l_{\nu_k} \geqq \lim_{h\to\infty} l_{\mu_h}.$$

Interchangig the partial sequences, we get

$$\lim_{h\to\infty} l_{\mu_h} \geqq \lim_{k\to\infty} l_{\nu_k},$$

hence $\lim l_{\nu_k} = \lim l_{\mu_h}$ and the sequence l_n is convergent.

I regard the proofs given here as being really elementary. From a more advanced point of view it may be remarked, that Lemma I and II are immediate consequences of the following theorem of G. H. Hardy[6]):

If the set $\beta_1, \beta_2, \ldots, \beta_n$ is majorized by $\alpha_1, \alpha_2, \ldots, \alpha_n$, then for any continous convex function $\Phi(x)$

$$\Phi(\beta_1)+\Phi(\beta_2)+\ldots+\Phi(\beta_n) > \Phi(\alpha_1)+\Phi(\alpha_2)+\ldots+\Phi(\alpha_n).$$

Under the conditions

$$\alpha_\mu > \beta_\nu > 0 \text{ for } \mu = 1, 2, \ldots, m;\ \nu = 1, 2, \ldots, n > m,$$
$$\alpha_\mu = 0 \quad \text{ for } \mu = m+1,\ m+2, \ldots, n,$$
$$\sum_{\mu=1}^{m} \alpha_\mu = \sum_{\nu=1}^{n} \beta_\nu$$

the set (β) is majorized by (α) and if we assume moreover that $\Phi(0) = 0$, then

$$(H) \quad \Phi(\beta_1)+\Phi(\beta_2)+\ldots+\Phi(\beta_n) > \Phi(\alpha_1)+\Phi(\alpha_2)+\ldots+\Phi(\alpha_m).$$

---

[5]) The length of any polygon, which contains the circle, is an upper bound of the length of all inscribed polygons and any expression of the form $\Pi(1-\beta_\nu)^{-1}$ is an upper bound of the products $\Pi(1+\alpha_\nu)$.

[6]) G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities* (1934), pp. 89–91.

Lemma I and II are special cases of this inequality for $\Phi(x) = \sin\frac{x}{2}$ $(\alpha_\mu = \widehat{A_\mu A_{\mu+1}}, \beta_\nu = \widehat{B_\nu B_{\nu+1}})$ resp. for $\Phi(x) = \log(1+x)$.

In virtue of (H), $\sum_{\nu=1}^{n} \Phi(\alpha_\nu)$ tends to a limit, if $\sum_{\nu=1}^{n} \alpha_\nu = C$ and $\max \alpha_\nu \to 0$, the limit being independent of the mode of subdivision of C. Hence the value of the limit is

$$\lim_{n\to\infty} n\,\Phi\left(\frac{C}{n}\right) = C\,\Phi'(+0).$$

The assumption $\Phi(x) = e^x - 1$ leads to another instructive application of (H). In this case

$$\sum_{\nu=1}^{n} e^{\alpha_\nu} - n \to C$$

if $\max \alpha_\nu \to 0$ and the α_ν are restricted by $\sum_{\nu=1}^{n} \alpha_\nu = C$ or, what is the same thing, by $\prod_{\nu=1}^{n} e^{\alpha_\nu} = e^C$. Denoting e^{α_ν} by $\frac{q_\nu}{q_{\nu-1}}$ and e^C by Γ, we have

$$\sum_{\nu=1}^{n} \frac{q_\nu - q_{\nu-1}}{q_{\nu-1}} \to \log \Gamma \tag{3}$$

if $\frac{q_n}{q_0} = \Gamma$ and $\frac{q_\nu}{q_{\nu-1}} \to 1$ $(\nu = 1, 2, \ldots, n)$.

To illustrate this result, take $q_\nu = n + \nu$ resp. $= \sqrt[n]{2^\nu}$, then $\Gamma = 2$ and (3) asserts that

$$\lim_{n\to\infty}\left(\frac{1}{n} + \frac{1}{n+1} + \ldots + \frac{1}{2n}\right) = \lim_{n\to\infty} n\left(\sqrt[n]{2} - 1\right) = \log 2.$$

(Received June 15, 1944.)