# On uniformly bounded linear transformations in Hilbert space.

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## §. 1.

In a paper aiming at a generalization of the theorem on spectral resolution of unitary transformations in HILBERT space[1]), E. R. LORCH has considered linear transformations $T$ in reflexive BANACH spaces[2]), uniformly bounded in the sense that the powers $T^n$ $(n=0, \pm 1, \pm 2, \ldots)$ have a common bound. All rotations fall under this type. LORCH points out that, even in HILBERT space, the spectral resolution of this type of transformations is investigated for the first time.

Now, in HILBERT space, Lorch's theorem is actually a consequence of the theorem on unitary transformations, owing to the fact that uniformly bounded transformations are *similar* to unitary ones. More precisely, we have

Theorem I. *Let $T$ be a linear transformation in Hilbert space $\mathfrak{H}$, such that its powers $T^n$ $(n=0, \pm 1, \pm 2, \ldots)$ are defined everywhere in $\mathfrak{H}$ and are uniformly bounded, i. e. $\|T^n\| \leqq k$ for some constant $k$. Then there exists a selfadjoint transformation $Q$, such that*

$$\frac{1}{k} I \leqq Q \leqq kI$$

*and $QTQ^{-1}$ is a unitary transformation.*

For one-parameter groups of transformations we have the corresponding

Theorem II *Let $T_s$ be a linear transformation in Hilbert space, depending on a real parameter $s$ $(-\infty < s < \infty)$, uniformly bounded,*

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1) E. R. LORCH, The integral representation of weakly almost periodic transformations in reflexive vector spaces, *Transactions American Math. Society*, **49** (1941), pp. 18—40.

2) I. e., a BANACH space which is the adjoint space of its adjoint space.

*i. e.* $\|T_s\| \leqq k$, *and possessing the group property*: $T_0 = I$; $T_s T_t = T_{s+t}$. *Then there exists a selfadjoint transformation* $Q$ *such that*

$$\frac{1}{k} I \leqq Q \leqq kI$$

*and* $Q T_s Q^{-1}$ *is unitary.*

§. 2.

We shall make use of the *generalized limit* due to MAZUR and BANACH[3]). This is a complex-valued functional $L(\xi(s))$, defined for all complex-valued bounded functions $\xi(s)$ of the positive real variable $s$, and enjoying the following properties:

1) $L(a\xi(s) + b\eta(s)) = aL(\xi(s)) + bL(\eta(s))$,
2) $L(\xi(s)) \geqq 0$ if $\xi(s) \geqq 0$,
3) $L(\xi(s+a)) = L(\xi(s))$ for all $a > 0$,
4) $L(1) = 1$.

Let us recall its construction.

Denote by $\Xi_c$ and $\Xi_r$ respectively the set of all complex-valued and the set of all real-valued bounded functions $\xi(s)$. If $L(\xi(s))$ has been defined already for $\xi(s) \in \Xi_r$ in such a way that 1)—4) are fulfilled, then we have only to put $L(\xi(s)) = L(\xi_1(s)) + iL(\xi_2(s))$ for $\xi(s) = \xi_1(s) + i\xi_2(s) \in \Xi_c$.

Now, for $\xi(s) \in \Xi_r$, define $L(\xi(s))$ in the following way.

Put $p(\xi) = \min_\sigma \left[ \overline{\lim_{s \to \infty}} \frac{1}{N} \sum_{k=1}^{N} \xi(\sigma_k + s) \right]$, $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N)$ running over all possible finite sets of positive real numbers. It may easily be verified that a) $p(c\xi) = cp(\xi)$ for $c \geqq 0$, b) $p(\xi + \eta) \leqq p(\xi) + p(\eta)$, c) $a \leqq p(\xi) \leqq b$ if $a \leqq \xi(s) \leqq b$ and d) $p(\xi(s) - \xi(s+a)) = p(\xi(s+a) - \xi(s)) = 0$.

Choose now a set $\{\xi\}$ of functions $\xi \in \Xi_r$, such that all elements of $\Xi_r$ may be expressed as finite linear combinations of the elements of the set $\{\xi\}$. Let us arrange it in a (transfinite) well-ordered sequence $\xi_0, \xi_1, \xi_2, \ldots, \xi_\omega, \ldots, \xi_\alpha, \ldots$ Denote by $G_\gamma$ ($\gamma$ being an arbitrary ordinal number $\geqq 1$) the set of all finite linear combinations of the $\xi_\alpha$ with $\alpha < \gamma$. We have $G_{\gamma_1} \leqq G_{\gamma_2}$ if $\gamma_1 < \gamma_2$, and there exists a first ordinal number $\Gamma$ for which $G_\Gamma = \Xi_r$.

The elements of $G_1$ are of the form $\xi = c\xi_0$ with a real $c$. Put

[3]) S. BANACH. *Théorie des opérations linéaires* (Warsaw, 1932), p. 33, Theorem 3.

$L(\xi)=L(c\xi_0)=cp(\xi_0)$; this is a linear functional in $G_1$ with the property: $L(\xi)\leqq p(\xi)$.[4])

Suppose that the linear functional $L(\xi)$ has been defined already for all $\xi\in G_\alpha$ with $\alpha<\gamma$, and that we have $L(\xi)\leqq p(\xi)$. We shall extend its definition to $G_\gamma$, still preserving the relation $L(\xi)\leqq p(\xi)$.

Suppose first that $\gamma$ has a predecessor, $\gamma=\beta+1$. If $\xi_\gamma\in G_\beta$, then $G_\gamma=G_\beta$ and we have nothing to do. If $\xi_\gamma$ does not belong to $G_\beta$, then the elements $\xi$ of $G_\gamma$ admit a unique representation $\xi=\eta+c\xi_\gamma$ with $\eta\in G_\beta$ and a real number $c$. If $\eta'$ and $\eta''\in G_\beta$, then

$$\begin{aligned} L(\eta'')-L(\eta')=L(\eta''-\eta')\leqq p(\eta''-\eta')= \\ =p((\eta''+\xi_\gamma)+(-\eta'-\xi_\gamma))\leqq p(\eta''+\xi_\gamma)+p(-\eta'-\xi_\gamma), \\ -p(-\eta'-\xi_\gamma)-L(\eta')\leqq p(\eta''+\xi_\gamma)-L(\eta''), \end{aligned}$$

consequently,

$$m=\max_{\eta\in G_\beta}[-p(-\eta-\xi_\gamma)-L(\eta)] \text{ and } M=\min_{\eta\in G_\beta}[p(\eta+\xi_\gamma)-L(\eta)]$$

are finite and $m\leqq M$. Choose a number $\mu$ between $m$ and $M$, and define $L(\xi)=L(\eta+c\xi_\gamma)=L(\eta)+c\mu$. This definition coincides on $G_\beta$ evidently with the old one and is such that $L(\xi)\leqq p(\xi)$.

If $\gamma$ is a limit number, then $G_\gamma=\sum_{\alpha<\gamma}G_\alpha$ and $L(\xi)$ is therefore already defined on $G_\gamma$.

As $G_\Gamma=\Xi_r$, we have defined $L(\xi)$ by transfinite recursion on the whole $\Xi_r$; $L(\xi)$ is a linear real-valued functional and such that $L(\xi)\leqq p(\xi)$. The properties 1)—4) follow now quite easily: 1) is fulfilled by linearity, 2): if $\xi(s)\geqq 0$ then $-L(\xi)=L(-\xi)\leqq p(-\xi)\leqq 0$, 3): $\pm L(\xi(s)-\xi(s+a))=L[\pm(\xi(s)-\xi(s+a))]\leqq p[\pm(\xi(s)-\xi(s+a))]=0$, 4): $L(1)\leqq p(1)=1$ and $-L(1)=L(-1)\leqq p(-1)=-1$ imply $L(1)=1$.

From this notion of generalized limit for functions it is easy to derive a corresponding one for bounded sequences $\xi(n)$ $(n=1, 2,\ldots)$ with the properties:

1) $L(a\xi(n)+b\eta(n))=aL(\xi(n))+bL(\eta(n))$,
2) $L(\xi(n))\geqq 0$ if $\xi(n)\geqq 0$,
3) $L(\xi(n_0+n))=L(\xi(n))$,
4) $L(1)=1$.

We have only to define $L(\xi(n))$ as $L(\xi(s))$, where $\xi(s)=\xi([s])$, $[s]$ denoting the greatest integer contained in the real number $s$.

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4) If $c\geqq 0$, then $L(\xi)=cp(\xi_0)=p(c\xi_0)=p(\xi)$; if $c<0$, then $L(\xi)= =cp(\xi_0)=-p(-c\xi_0)=-p(-\xi)=-p(-\xi)+p(\xi-\xi)\leqq -p(-\xi)+p(\xi)+ +p(-\xi)=p(\xi)$.

## §. 3.

Let us now go onto the proof of Theorem I.

Let $f$ and $g$ be elements of $\mathfrak{R}$. The sequence $\xi(n) = (T^n f, T^n g)$ $(n=0,1,2,\ldots)$ being bounded, $|\xi(n)| \leqq k^2 \|f\| \, \|g\|$, we may put

$$\langle f,g\rangle = L(T^n f, T^n g).$$

By property 1) of the generalized limit, we have

$$\langle a_1 f_1 + a_2 f_2, b_1 g_1 + b_2 g_2 \rangle =$$
$$= a_1 \bar{b}_1 \langle f_1, g_1\rangle + a_1 \bar{b}_2 \langle f_1, g_2\rangle + a_2 \bar{b}_1 \langle f_2, g_1\rangle + a_2 \bar{b}_2 \langle f_2, g_2\rangle,$$

i. e., $\langle f, g\rangle$ is a hermitian bilinear form of the variable elements $f$ and $g$. Furthermore, the inequalities

$$\frac{1}{k} \leqq \frac{\|T^n f\|}{\|T^{-n} T^n f\|} = \frac{\|T^n f\|}{\|f\|} \leqq k$$

imply, by the properties 1., 2. and 4., that

$$\frac{1}{k^2} \|f\|^2 \leqq \langle f,f\rangle \leqq k^2 \|f\|^2. \tag{1}$$

By a known theorem on bounded hermitian bilinear forms[5]), there exists a selfadjoint transformation $A$ such that $\langle f,g\rangle = (Af, g)$. We have, by (1),

$$\frac{1}{k^2} I \leqq A \leqq k^2 I, \tag{2}$$

and by property 3),

$$(ATf, Tg) = L(T^{n+1} f, T^{n+1} g) = L(T^n f, T^n g) = (Af, g),$$

i. e.

$$T^* A T = A. \tag{3}$$

Let $Q$ be the positive selfadjoint square-root of $A$; we have, as a consequence of (2):

$$\frac{1}{k} I \leqq Q \leqq kI,$$
$$\frac{1}{k} I \leqq Q^{-1} \leqq kI.$$

It follows from (3) that

$$Q^{-1}(T^* Q Q T) Q^{-1} = Q^{-1}(QQ) Q^{-1} = I,$$
$$(QTQ^{-1})^* (QTQ^{-1}) = I.$$

Thus, $U = QTQ^{-1}$ is isometric. As it admits an inverse, namely $U^{-1} = QT^{-1}Q^{-1}$, it is also unitary. This completes the proof of Theorem I.

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[5]) See e. g. M. H. Stone, *Linear transformations in Hilbert space* (New York, 1932), p. 63, Theorem 2. 28.

The demonstration of Theorem II runs along the same lines. The transformation $A$ has to be defined in this case by

$$(Af, g) = L(T_s f, T_s g),$$

where $L$ denotes the generalized limit for functions.

## §. 4.

The above proof being based on the notion of generalized limit, depends, as we have seen in §. 2, on Zermelo's well-ordering theorem. It is therefore worthy to observe that — at least in the case of a *separable* HILBERT space — Zermelo's theorem may be avoided, owing to the fact that, in this case, we do not need to define $L(\xi)$ for *all* bounded functions (or sequences), but only on a certain linear manifold of such functions (or sequences), determined by a denumerable subset of its elements.

Let us consider first the case of Theorem I.

Choose a (denumerable) complete system $f_0, f_1, f_2, \ldots$ of elements of the separable space $\mathfrak{R}$. Arrange the sequences

$$\xi^{(ijk)}(n) = (T^{n+i} f_j, T^{n+i} f_k) \qquad (i, j, k = 0, 1, 2, \ldots)$$

in a single row:

$$\xi_0(n), \xi_1(n), \xi_2(n), \ldots$$

Let us construct the ascending sequence of linear manifolds $G_1 \subseteq G_2 \subseteq \subseteq G_3 \subseteq \ldots$ and $G_\omega = \prod_{\nu=1}^{\infty} G_\nu$, and let us then define the functional $L(\xi(n))$ for all sequences $\xi(n) \in G_\omega$ in the same way as in § 2. If $\mathfrak{R}'$ denotes the linear manifold of all finite linear combinations of the elements $T^i f_k$ $(i, k = 0, 1, 2, \ldots)$, then $(T^{j+n} f, T^{j+n} g) \in G_\omega$ for any couple $f, g$ of elements of $\mathfrak{R}'$ and any integer $j \geqq 0$. Thus

$$\langle f, g \rangle = L(T^n f, T^n g)$$

is defined for all $f, g \in \mathfrak{R}'$, is a bounded hermitian bilinear form and such that

$$\langle f, g \rangle = \langle Tf, Tg \rangle. \tag{4}$$

As $\langle f, g \rangle$ is bounded, its definition may be extended to the whole space $\mathfrak{R}$, the relation (4) holding for arbitrary elements $f, g$ of $\mathfrak{R}$, by the continuity of $T$. The demonstration achieves as in §. 3.

Now we pass to Theorem II. In addition to its hypothesis, *let us suppose also that $T_s$ depends continuously on $s$*, i. e. $T_s f \to T_t f$ if $s \to t$. Let the system $f_0, f_1, f_2, \ldots$ be as above. The set of functions

$(T_{r+s}f_j, T_{r+s}f_k)$ of $s$ (with integer $j,k$ and rational $r$) being denumerable, may be arranged in a sequence:

$$\xi_0(s), \xi_1(s), \xi_2(s), \ldots$$

Define the functional $L(\xi)$ on the linear manifold $G_\omega$ of all finite linear combinations of the $\xi_\nu(s)$ in the same way as in §. 2. If $\mathfrak{R}'$ denotes the linear manifold in $\mathfrak{R}$ formed by the finite linear combinations of the elements $T_r f_k$ ($r$ rational, $k$ integer), then the function $(T_{r+s}f, T_{r+s}g)$ of the variable $s$ belongs to $G_\omega$ for any couple $f, g \in \mathfrak{R}'$. The form

$$\langle f,g\rangle = L(T_s f, T_s g)$$

is thus defined and we have

$$\langle f,g\rangle = \langle T_r f, T_r g\rangle. \tag{5}$$

The definition of the bounded form $\langle f,g\rangle$ may be extended to the whole space $\mathfrak{R}$, the relation (5) holding its validity, because $T_r$ is a continuous transformation. Thus we have $\langle f,g\rangle = (Af,g)$ and $(Af,g) = (AT_r f, T_r g)$ for all rational numbers $r$. As $T_s$ depends, by hypothesis, continuously on $s$, we have $(Af,g) = (AT_s f, T_s g)$ also for irrational $s$. That is, $A = T_s^* A T_s$, and the proof achieves in the same way as in §. 3.

*(Received August 20, 1945.)*