A theorem on convex curves.

By I. S. GÁL in Budapest.

In his foregoing paper, A. RÉNYI has proved by analytical methods, besides other results, the following inequality for convex curves:

$$A \leqq \varrho P\left(\frac{\varrho}{2}\right) \tag{1}$$

where A deno the area of the curve, ϱ the radius of the greatest inscribable circle and $P\left(\frac{\varrho}{2}\right)$ the periphery of the internal parallel curve at the distance $\frac{\varrho}{2}$.

In what follows an elementary proof of this inequality shall be given. We prove (1) for polygons; for general convex curves it follows by passing to the limit.

Our proof is based on the following

Lemma. If the convex polygon $(\ldots, B, C, D, E, \ldots)$ is augmented by prolonging the sides BC and DE until they meet at O, the increase of the left hand side of (1) is greater than that of the right hand side.

Proof. Two cases have to be distinguished. If the radius r of the circle externally tangential at CD to the triangle OCD is less than $\frac{\varrho}{2}$, the right hand side of (1) does not change at all, and thus the statement of our lemma is obvious. If however $r > \frac{\varrho}{2}$, we draw $B'C'$ $C'D'$ and $D'E'$ parallel to, and at the distance $\frac{\varrho}{2}$ from BC, CD and DE respectively. The point of intersection of the prolongations of $B'C'$ and $D'E'$ shall be denoted by O'. The triangles CDO and $C'D'O'$ are similar. The factor of proportionality is

$$\frac{C'D'}{CD} = 1 - \frac{\varrho}{2r}. \tag{2}$$

By passing from the polygon $(\ldots, B, C, D, E, \ldots)$ to the polygon $(\ldots, B, O, D, \ldots)$, the left side of (1) is increased by

$$\Delta[A] = \frac{r}{2}(CO + OD - CD), \tag{3}$$

its right side by

$$\Delta\left[\varrho P\left(\frac{\varrho}{2}\right)\right] = \varrho\,(C'O' + O'D' - C'D'). \tag{4}$$

Owing to (2), we get from (4)

$$\Delta\left[\varrho P\left(\frac{\varrho}{2}\right)\right] = \varrho\left(1 - \frac{\varrho}{2r}\right)(CO + OD - CD). \tag{5}$$

Comparing (3) with (5), the statement of our lemma is reduced to the inequality

$$\varrho\left(1 - \frac{\varrho}{2r}\right) \leqq \frac{r}{2}, \text{ i. e. } 2r\varrho \leqq r^2 + \varrho^2.$$

Thus our lemma is proved.

Now the inequality (1) follows easily. In fact, removing the sides of the given convex polygon step by step, by prolonging two adjacent sides conforming to the above lemma, the difference $A - \varrho P\left(\frac{\varrho}{2}\right)$ gets increased. As it is well known, the polygon has either three sides touching the inscribed circle in three points not lying on a half-circle, or two parallel sides touching the inscribed circle. Taking care that these sides shall not be removed, after a finite number of steps the polygon gets transformed to a triangle or a trapezium (eventually parallelogram), having the same inscribed circle. Now, for a triangle we have obviously $A = \varrho P\left(\frac{\varrho}{2}\right)$; the same holds for a trapezium circumscribed to a circle. In the general case, replace the trapezium by another having the same inscribed circle, all sides of which, parallel to the corresponding sides of the original trapezium, touch that circle. Performing this operation, we have obviously

$$\Delta A = \Delta\left[\varrho P\left(\frac{\varrho}{2}\right)\right];$$

thus, we have $A = \varrho P\left(\frac{\varrho}{2}\right)$ in this case too. Consequently, $A - \varrho P\left(\frac{\varrho}{2}\right)$ has not been positive for the original polygon, q. e. d.

(Received February 27, 1946.)