On quasi-primary ideals.

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In a domain of integrity with unit element there are known four, generally distinct representations of an ideal: the representation as intersection of (1) irreducible ideals, (2) primary ideals belonging to different prime ideals, (3) relative-prime-irreducible ideals, and finally (4) ideals without common divisor[1]). In the present paper I give a fifth representation of an ideal as intersection of quasi-primary ideals.

In the first part, after the definition of the quasi-primary ideals, their chief properties will be shown. Although the notion of the quasi-primary ideal is a generalization of that of the primary ideal, yet most of the theorems concerning the primary ideals remain true even for quasi-primary ideals; moreover, these theorems are characteristic for the quasi-primary ideals.

In the second part, it will be proved by making use of the maximal condition that every ideal is representable as intersection of a finite number of quasi-primary ideals; furthermore, the number of the components in a shortest representation as well as the prime ideals belonging to the quasi-primary ideals are uniquely determined. The proof based upon the notion of the radical is more simple than that of the corresponding theorem on primary ideals. Indeed, we do not need complete induction, because every prime ideal belonging to the quasi-primary ideals in a shortest representation is maximal. It will be shown that among the four Noetherian representations quoted above a shortest representation by means of quasi-primary ideals always can be fitted, in a certain sense, between (2) and (3).

Finally, in the third part, I examine how the quasi-primary ideals are applicable to rings of algebraic numbers and to the theory of polynomial ideals. It will be seen that the introduction of the quasi-primary ideals seems to be very useful especially in the last case.

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[1]) See E. NOETHER, Idealtheorie in Ringbereichen, *Math. Annalen,* 83 (1921), pp. 24--66.

§ 1. The quasi-primary ideals.

Let $\mathfrak{R}$ be a domain of integrity[2]) with unit element in which the maximal condition is satisfied. (The elements of $\mathfrak{R}$ shall be denoted with Greek letters.)

Definition 1. *The ideal $\mathfrak{q}$ of the ring $\mathfrak{R}$ is quasi-primary, if the congruence $\alpha\beta \equiv 0\,(\mathfrak{q})$ implies that among the powers α^r and β^s there exists one which is $\equiv 0\,(\mathfrak{q})$; i. e. $\alpha\beta \equiv 0\,(\mathfrak{q})$ and $\alpha^r \not\equiv 0\,(\mathfrak{q})$ for every r imply the existence of an s such that $\beta^s \equiv 0\,(\mathfrak{q})$.*

The definition can be expressed also in the following form: $\mathfrak{q}$ is quasi-primary if at least one of two conjugate divisors of zero[3]) of the residue-class ring $\mathfrak{R}/\mathfrak{q}$ is nilpotent[4]).

Evidently, the definition of the quasi-primary ideals is more symmetrical than that of the primary ideals; furthermore, it is evident that every primary ideal is at the same time quasi-primary too; but it will be seen that the conversion is not always true: there exist quasi-primary ideals which are not primary.

From the maximal condition we obtain that every quasi-primary ideal $\mathfrak{q}$ is also *strong*; i. e. if $\mathfrak{a}\mathfrak{b} \equiv 0\,(\mathfrak{q})$, but $\mathfrak{a}^r \not\equiv 0\,(\mathfrak{q})$ for every r, then there exists an s so that $\mathfrak{b}^s \equiv 0\,(\mathfrak{q})$. The proof runs as follows.

The ideals $\mathfrak{a}$ and $\mathfrak{b}$ have finite bases (this follows immediately from the maximal condition): $\mathfrak{a} = (\alpha_1, \ldots, \alpha_n)$ and $\mathfrak{b} = (\beta_1, \ldots, \beta_m)$ respectively. Now, for every r we have $\mathfrak{a}^r \not\equiv 0\,(\mathfrak{q})$, thus there is an α which has no power in the ideal $\mathfrak{q}$. Indeed, supposing we should have $\alpha_i^{r_i} \equiv 0\,(\mathfrak{q})$ for $i = 1, \ldots, n$, then choose $r = \sum_{i=1}^{n} (r_i - 1) + 1$ and so $\mathfrak{a}^r = (\ldots, \alpha_1^{t_1} \ldots \alpha_n^{t_n}, \ldots) \equiv 0\,(\mathfrak{q})$, because at least for one i we have $t_i \geqq r_i$. This is a contradiction to our hypothesis that $\mathfrak{a}^r \not\equiv 0\,(\mathfrak{q})$ for every r. Now, for this α_j by hypothesis $\alpha_j \beta_l \equiv 0\,(\mathfrak{q})$ $(l = 1, \ldots, m)$; but $\alpha_j^t \not\equiv 0\,(\mathfrak{q})$ for every t, consequently there exists an s_l such that $\beta_l^{s_l} \equiv 0\,(\mathfrak{q})$ $(l = 1, \ldots, m)$. Let $s = \sum_{l=1}^{m} (s_l - 1) + 1$, then $\mathfrak{b}^s \equiv 0\,(\mathfrak{q})$, q. e. d.

It is well known that the elements which have a power in the ideal $\mathfrak{a}$, form an ideal, the radical[5]) of $\mathfrak{a}$. Now we shall prove that the quasi-primary ideals can be defined by means of their radical as follows.

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[2]) I. e., a commutative ring without divisors of zero.

[3]) $a \neq 0$ and $b \neq 0$ are conjugate divisors of zero if $ab = 0$.

[4]) I. e., *one of its powers is zero.*

[5]) For the notion of the radical see W. Krull, *Idealtheorie* (Berlin, 1935), p. 6. If $\mathfrak{r}$ is the radical of $\mathfrak{a}$, then of course $\mathfrak{a} \subseteqq \mathfrak{r}$.

Definition 2. *An ideal is quasi-primary if its radical is prime.*

Let $\mathfrak{p}$ be the radical of the quasi-primary ideal $\mathfrak{q}$. If $\alpha\beta \equiv 0\,(\mathfrak{p})$ but $\alpha \not\equiv 0\,(\mathfrak{p})$, then there exists an integer t for which $\alpha^t\beta^t \equiv 0\,(\mathfrak{q})$, but $\alpha^{tr} \not\equiv 0\,(\mathfrak{q})$ for every r; hence $\beta^{ts} \equiv 0\,(\mathfrak{q})$; i. e. $\beta \equiv 0\,(\mathfrak{p})$. Therefore $\mathfrak{p}$ is prime in fact. $\mathfrak{p}$ is the prime ideal belonging to the quasi-primary ideal $\mathfrak{q}$.

Conversely, let the prime ideal $\mathfrak{p}$ be the radical of the ideal $\mathfrak{q}$. If $\alpha\beta \equiv 0\,(\mathfrak{q})$ but $\alpha^r \not\equiv 0\,(\mathfrak{q})$ for every r, then $\alpha\beta \equiv 0\,(\mathfrak{p})$, but $\alpha \not\equiv 0\,(\mathfrak{p})$ and this leads to the congruence $\beta \equiv 0\,(\mathfrak{p})$. Consequently $\beta^s \equiv 0\,(\mathfrak{q})$, that is to say, $\mathfrak{q}$ is in fact quasi-primary. $\mathfrak{q}$ is a quasi-primary ideal belonging to the prime ideal $\mathfrak{p}$.

Of course, the primary ideals have also the property that their radical is prime, but an ideal, the radical of which is prime, is not necessarily primary. An instance is given by VAN DER WAERDEN[6]): in the ring of the polynomials $a_0 + a_1x + \ldots + a_nx^n$, where the a_i are rational integers and a_1 is divisible by 3, the ideal $\mathfrak{q} = (9x^2, 3x^3, x^4, x^5, x^6)$ is not primary (because $9x^2 \equiv 0\,(\mathfrak{q})$ and $9^r \not\equiv 0\,(\mathfrak{q})$ for every r, but $x^2 \not\equiv 0\,(\mathfrak{q})$), though its radical $\mathfrak{p} = (3x, x^2, x^3)$ is prime.

As the radical of $\mathfrak{a}$ and that of $\mathfrak{a}^t$ are identical, $\mathfrak{a}^t$ is quasi-primary if and only if $\mathfrak{a}$ is quasi-primary too. Thus every power of a prime ideal as well as that of a primary or a quasi-primary ideal is quasi-primary; but a power of a prime ideal must not be primary; see the above instance of VAN DER WAERDEN where $\mathfrak{q} = \mathfrak{p}^2$.

From the second definition of the quasi-primary ideals it is evident that the least common multiple of the quasi-primary ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$,

$$\mathfrak{m} = [\mathfrak{q}_1, \ldots, \mathfrak{q}_r]$$

is quasi-primary if and only if the least common multiple of the prime ideals $\mathfrak{p}_i$ belonging to the quasi-primary ideals $\mathfrak{q}_i$,

$$\mathfrak{r} = [\mathfrak{p}_1, \ldots, \mathfrak{p}_r]$$

is prime[7]). The intersection of a finite number of prime ideals is prime only if one of them is a multiple of the others[8]); thus we have the following result:

Theorem 1. *The ideal $\mathfrak{m} = [\mathfrak{q}_1, \ldots, \mathfrak{q}_r]$ is quasi-primary if and only if among the prime ideals $\mathfrak{p}_i$ there is a $\mathfrak{p}_k$ such that $\mathfrak{p}_k \equiv 0\,(\mathfrak{p}_i)$ $(i = 1, \ldots, r)$.*

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6) B L. VAN DER WAERDEN, *Moderne Algebra*, vol. II (Berlin, 1940), p. 27.

7) The radical of the l. c. m. of ideals is the l. c. m. of the radicals.

8) If $[\mathfrak{p}_1, \ldots, \mathfrak{p}_r] = \mathfrak{p}$ is prime, then $\mathfrak{p}_1 \ldots \mathfrak{p}_r \equiv 0(\mathfrak{p})$, hence $\mathfrak{p}_k \equiv 0(\mathfrak{p})$ for a k, in consequence of the prime-property of $\mathfrak{p}$. Thus $\mathfrak{p}_k = \mathfrak{p} \subseteqq \mathfrak{p}_i$.

By making use of the maximal condition we can prove the

Theorem 2. *A power of the prime ideal* $\mathfrak{p}$ *belonging to the quasi-primary ideal* $\mathfrak{q}$ *is a multiple of* $\mathfrak{q}$:

$$\mathfrak{p}^r \equiv 0\,(\mathfrak{q}).$$

Taking $\mathfrak{p} = (\gamma_1, \ldots, \gamma_p)$, the basis-elements γ_q have a power in the ideal $\mathfrak{q}$: $\gamma_q^{r_q} \equiv 0\,(\mathfrak{q})$. Let r have the value $r = \sum_{q=1}^{p} (r_q - 1) + 1$. Now

$$\mathfrak{p}^r = (\ldots, \gamma_1^{s_1} \ldots \gamma_p^{s_p}, \ldots) \equiv 0\,(\mathfrak{q})$$

$\left(\sum_{q=1}^{p} s_q = r\right)$, because at least for one of the subscripts q we have $s_q \geqq r_q$.

The least r for which $\mathfrak{p}^r \equiv 0\,(\mathfrak{q})$, is called the *exponent* of $\mathfrak{q}$.

We see that the prime ideal $\mathfrak{p}$ belonging to the quasi-primary ideal $\mathfrak{q}$ has the property:

(1) $$\mathfrak{p}^r \subseteqq \mathfrak{q} \subseteqq \mathfrak{p}.$$

This relation is characteristic for $\mathfrak{p}$: if $\mathfrak{q}$ is quasi-primary, $\mathfrak{p}'$ is prime and $\mathfrak{p}'^{r'} \subseteqq \mathfrak{q} \subseteqq \mathfrak{p}'$, then $\mathfrak{p}'$ is the radical of $\mathfrak{q}$. Indeed, if $\mathfrak{p}$ is the radical of $\mathfrak{q}$, we have

$$\mathfrak{p}^r \subseteqq \mathfrak{q} \subseteqq \mathfrak{p}', \text{ hence } \mathfrak{p} \subseteqq \mathfrak{p}'$$

and

$$\mathfrak{p}'^{r'} \subseteqq \mathfrak{q} \subseteqq \mathfrak{p}, \text{ hence } \mathfrak{p}' \subseteqq \mathfrak{p},$$

whence $\mathfrak{p}' = \mathfrak{p}$ is obtained.

The relation (1) is characteristic even for quasi-primary ideals if $\mathfrak{p}$ is prime and $\mathfrak{p}^r \subseteqq \mathfrak{q} \subseteqq \mathfrak{p}$, then $\mathfrak{q}$ is quasi-primary. Moreover, we can prove the

Theorem 3. *If* $\mathfrak{q}_1$ *and* $\mathfrak{q}_2$ *are quasi-primary ideals belonging to the same prime ideal* $\mathfrak{p}$ *and* $\mathfrak{q}_1 \subset \mathfrak{q}' \subset \mathfrak{q}_2$, *then* $\mathfrak{q}'$ *is also quasi-primary belonging to the prime ideal* $\mathfrak{p}$.[9])

The radical $\mathfrak{p}'$ of $\mathfrak{q}'$ is divisor of the radical $\mathfrak{p}$ of $\mathfrak{q}_1$, because the elements of $\mathfrak{p}$ have a power in $\mathfrak{q}_1$ and so a fortiori in $\mathfrak{q}'$, i. e. $\mathfrak{p} \subseteqq \mathfrak{p}'$; similarly, the elements of $\mathfrak{p}'$ have a power in $\mathfrak{q}_2$, that is $\mathfrak{p}' \subseteqq \mathfrak{p}$. Consequently, we have $\mathfrak{p}' = \mathfrak{p}$. This means, the radical of $\mathfrak{q}'$ is prime; thus $\mathfrak{q}'$ is in fact quasi-primary.

It follows from theorem 3 that a relation $\mathfrak{q}^t \subset \mathfrak{q}' \subset \mathfrak{q}$ implies the quasi-primary property of $\mathfrak{q}'$ provided that $\mathfrak{q}$ is quasi-primary.

It is interesting to note that the quasi-primary ideals can be characterized in rings with maximal condition by theorems 2 and 3 as

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9) This theorem is not true for primary ideals. It does not follow even from the hypothesis $\mathfrak{p}^r \subseteqq \mathfrak{q}' \subseteqq \mathfrak{p}$ that $\mathfrak{q}'$ is primary. See the above example of VAN DER WAERDEN; $\mathfrak{p}^2 = \mathfrak{q} \subset \mathfrak{p}$.

follows. *A quasi-primary ideal is either a power of a prime ideal or an intermediate ideal between two powers of one and the same prime ideal.*

The following theorem has no analogue in the theory of the primary ideals[10]).

Theorem 4. *If $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are quasi-primary ideals belonging to the prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ respectively, and $\mathfrak{p}_1 \subseteqq \mathfrak{p}_2$, then $\mathfrak{q} = \mathfrak{q}_1 \mathfrak{q}_2$ is also quasi-primary belonging to the prime ideal $\mathfrak{p}_1$.*

Taking $\mathfrak{p}_1^{r_1} \subseteqq \mathfrak{q}_1 \subseteqq \mathfrak{p}_1$ and $\mathfrak{p}_2^{r_2} \subseteqq \mathfrak{q}_2 \subseteqq \mathfrak{p}_2$, furthermore observing the evident facts that $\mathfrak{p}_1^{r_1+r_2} \subseteqq \mathfrak{p}_1^{r_1} \mathfrak{p}_2^{r_2}$ and $\mathfrak{p}_1 \mathfrak{p}_2 \subset \mathfrak{p}_1$, we have for $\mathfrak{q} = \mathfrak{q}_1 \mathfrak{q}_2$ the relation

$$\mathfrak{p}_1^{r_1+r_2} \subseteqq \mathfrak{p}_1^{r_1} \mathfrak{p}_2^{r_2} \subseteqq \mathfrak{q} \subseteqq \mathfrak{p}_1 \mathfrak{p}_2 \subset \mathfrak{p}_1 .$$

Hence we obtain by theorem 3 that $\mathfrak{q}$ is quasi-primary and indeed, its radical is $\mathfrak{p}_1$.

It is also worth remarking that the hypothesis of theorem 4 also implies that the greatest common divisor of $\mathfrak{q}_1$ and $\mathfrak{q}_2$, $(\mathfrak{q}_1, \mathfrak{q}_2)$, is quasi-primary belonging to the prime ideal $\mathfrak{p}_2$. In fact,

$$\mathfrak{q}_2 \subseteqq (\mathfrak{q}_1, \mathfrak{q}_2) \subseteqq (\mathfrak{p}_1, \mathfrak{p}_2) = \mathfrak{p}_2$$

and by theorem 3, $(\mathfrak{q}_1, \mathfrak{q}_2)$ is quasi-primary.

§ 2. The representation of an ideal by means of quasi-primary ideals.

An ideal is called *reducible*, if it is the least common multiple of two of its proper divisors, and is called *irreducible*, if it is not reducible.

E. NOETHER[1]) proved that every ideal is the intersection of a finite number of irreducible ideals, if the maximal condition is satisfied. Every irreducible ideal is primary and so a fortiori quasi-primary; hence we have

Theorem 5. *Every ideal is the intersection of a finite number of quasi-primary ideals.*

A representation of the ideal $\mathfrak{a}$ as least common multiple of quasi-primary ideals,

$$\mathfrak{a} = [\mathfrak{q}_1, \ldots, \mathfrak{q}_s],$$

is *shortest*, if none of the $\mathfrak{q}_i$ can be omitted and none of the intersections $[\mathfrak{q}_{i_1}, \ldots, \mathfrak{q}_{i_p}]$ $(p > 1)$ is quasi-primary.

Now omit the superfluous $\mathfrak{q}_i$ of a given quasi-primary representation of $\mathfrak{a}$ and contract the quasi-primary ideals belonging to the

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[10]) We can refer to the above example of VAN DER WAERDEN; $\mathfrak{p}\mathfrak{p}$ is not primary.

non-minimal prime ideals of $\mathfrak{a}$ with a quasi-primary ideal belonging to such a minimal prime ideal of $\mathfrak{a}$ which has the non-minimal prime ideal as its divisor. In this way always quasi-primary ideals are obtained. Proceeding thus, after a finite set of contractions we get a shortest representation of $\mathfrak{a}$ as intersection of quasi-primary ideals.

The prime ideals belonging to the quasi-primary ideals which occur in a shortest representation of $\mathfrak{a}$ have the property that none of them is either a multiple or a divisor of another of them. Indeed, in the opposite case a contraction is possible.

For two shortest representations of $\mathfrak{a}$ the following theorem holds:

Theorem 6. *Supposing that* $\mathfrak{a}=[\mathfrak{q}_1,\ldots,\mathfrak{q}_s]=[\mathfrak{q}'_1,\ldots,\mathfrak{q}'_r]$ *are two shortest representations of the ideal* $\mathfrak{a}$ *as intersection of quasi-primary ideals; then* $s=r$ *and the prime ideals* $\mathfrak{p}_i$ *belonging to the quasi-primary ideals* $\mathfrak{q}_i$ *must be, without regard to their order, identical to the prime ideals* $\mathfrak{p}'$ *belonging to the quasi-primary ideals* $\mathfrak{q}'_j$.

This theorem is our main result; it can be proved as follows.

The radical $\mathfrak{r}$ of $\mathfrak{a}$ is of the form

$$\mathfrak{r}=[\mathfrak{p}_1,\ldots,\mathfrak{p}_s]$$

and at the same time of the form

$$\mathfrak{r}=[\mathfrak{p}'_1,\ldots,\mathfrak{p}'_r].$$

The identity of these representations is to be shown.

From these two forms of $\mathfrak{r}$ no prime ideal can be omitted, because by hypothesis the representations of $\mathfrak{a}$ are shortest ones. Indeed, in the two representations of $\mathfrak{a}$ the prime ideals belonging to the quasi-primary ideals are the different minimal prime ideals of $\mathfrak{a}$; thus e. g.

$$[\mathfrak{p}_{i_1},\ldots,\mathfrak{p}_{i_p}]\equiv 0 \quad ([\mathfrak{p}_{i'_1},\ldots,\mathfrak{p}_{i'_q}])$$

is impossible if $i_{p'}\neq i'_{q'}$ for every $p'=1,\ldots,p$ and every $q'=1,\ldots,q$. For now $\mathfrak{p}_{i_1}\ldots\mathfrak{p}_{i_p}\equiv 0 \quad (\mathfrak{p}_{i'_1})$ and so, say, $\mathfrak{p}_{i_1}\equiv 0 \quad (\mathfrak{p}_{i'_1})$, i. e. because of $\mathfrak{p}_{i_1}\neq\mathfrak{p}_{i'_1}$, $\mathfrak{p}_{i'_1}$ would not be minimal. This is contradiction to the hypothesis that the representations of $\mathfrak{a}$ are shortest ones.

Now, if in the above two forms of $\mathfrak{r}$ the prime ideals would be different, then at least in one of these forms a maximal prime ideal would be found which occurs only in one of the forms. Indeed, let $\mathfrak{p}$ be such a prime ideal e. g. in the first representation of $\mathfrak{r}$ which does not occur in the second one. The other prime ideals of the first representation of $\mathfrak{r}$ are not divisors of $\mathfrak{p}$. If $\mathfrak{p}$ has no divisor even among the prime ideals of the second representation of $\mathfrak{r}$, then already we have found a suitable prime ideal. If $\mathfrak{p}$ has a divisor $\mathfrak{p}'$ among the $\mathfrak{p}'_j$, then $\mathfrak{p}'$ must be a proper divisor of $\mathfrak{p}$ because of $\mathfrak{p}$ and $\mathfrak{p}'$ being different ideals. This $\mathfrak{p}'$ has a divisor neither among the $\mathfrak{p}'_j$ nor among the $\mathfrak{p}_i$. The first part of our last statement is evident, the second one is also clear, because the divisor of $\mathfrak{p}'$ would be a divisor of $\mathfrak{p}$ and such an ideal does not exist among the $\mathfrak{p}_i$. This means, if the prime ideals of the two forms of $\mathfrak{r}$ are different, then there exists a maximal prime ideal, e. g. $\mathfrak{p}_1$, which occurs only in one of the two forms of $\mathfrak{r}$.

We consider this $\mathfrak{p}_1$ and form the ideal-quotient $\mathfrak{r} : \mathfrak{p}_1$:[11])

$$\mathfrak{r} : \mathfrak{p}_1 = [\mathfrak{p}_1 : \mathfrak{p}_1, \ldots, \mathfrak{p}_s : \mathfrak{p}_1] = [\mathfrak{p}'_1 : \mathfrak{p}_1, \ldots, \mathfrak{p}'_r : \mathfrak{p}_1].$$

If $\mathfrak{p}$ is prime and $\mathfrak{p}_1 \not\equiv 0\,(\mathfrak{p})$, then $\gamma\,\mathfrak{p}_1 \equiv 0\,(\mathfrak{p})$ implies $\gamma \equiv 0\,(\mathfrak{p})$, this means $\mathfrak{p} : \mathfrak{p}_1 = \mathfrak{p}$ if $\mathfrak{p}_1 \not\equiv 0\,(\mathfrak{p})$. Therefore,

$$\mathfrak{r} : \mathfrak{p}_1 = [\mathfrak{o}, \mathfrak{p}_2, \ldots, \mathfrak{p}_s] = [\mathfrak{p}'_1, \ldots, \mathfrak{p}'_r] = \mathfrak{r}$$

($\mathfrak{o}$ is the unit ideal of $\mathfrak{R}$). Consequently, $\mathfrak{p}_1$ is superfluous in the first form of $\mathfrak{r}$. It follows from this contradiction that the prime ideals are the same in both forms of $\mathfrak{r}$.

Thus we have proved theorem 6.

We shall give also another proof of our theorem 6 by making use of the similar theorem concerning the primary ideals[12]). The representation of $\mathfrak{a}$ as intersection of primary ideals be already known:

$$\mathfrak{a} = [\mathfrak{q}_1^*, \ldots, \mathfrak{q}_n^*].$$

This representation can be considered as a representation of $\mathfrak{a}$ as intersection of quasi-primary ideals. From this representation a shortest quasi-primary representation is obtained by contracting certain primary ideals. This process is similar to that at the beginning of this section and so we need not repeat it. From the uniquely determined prime ideals belonging to the primary ideals in the primary representation of $\mathfrak{a}$, only the minimal prime ideals of $\mathfrak{a}$, determined also uniquely, remain, as the prime ideals belonging to the quasi-primary ideals in a shortest representation. Thus our theorem 6 is proved again[12]).

From the representation of $\mathfrak{a}$ as intersection of primary ideals the representation as intersection of relative-prime-irreducible ideals is obtained as follows[13]). Starting from a primary component of $\mathfrak{a}$, we take all the primary components belonging to such a prime ideal which is multiple or divisor of the prime ideal belonging to the original primary component; then take all the primary components, the prime ideal of which is multiple or divisor of a prime ideal belonging to a primary component obtained formerly, etc. The intersection of all the primary components of $\mathfrak{a}$ obtained in this way is a component of $\mathfrak{a}$ in its representation as intersection of relative-prime-irreducible ideals.

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[11]) For the notion of the ideal-quotient and its properties, see e. g. B. L. van der Waerden, loc. cit.[6]), p. 24.

[12]) See E. Noether, loc. cit.[1]), p. 44, and B. L. van der Waerden, loc. cit.[6]), p 35. — Our second proof is not of general validity, it will do only for quasi-primary representations got by contracting primary components!

[13]) This process is given by E. Noether; see her paper [1]), pp. 47–48.

From the preceding process it is evident that the primary ideals belonging to the same quasi-primary component of $\mathfrak{a}$ belong also to the same component of $\mathfrak{a}$ in its representation as intersection of relative-prime-irreducible ideals. Therefore, the representation as least common multiple of relative-prime-irreducible ideals is obtained by contracting certain quasi-primary ideals of a shortest quasi-primary representation. So, it is shown that there exists a representation of $\mathfrak{a}$ as intersection of quasi-primary ideals as an intermediate one between the representations by means of primary and relative-prime-irreducible ideals.

Now, in order to illustrate that the representation of an ideal as intersection of quasi-primary ideals differs, in general, both from the primary representation and from the relative-prime-irreducible representation, let us consider the ring of the polynomials in x and y with coefficients in a commutative field.

In this ring the ideal

$$\mathfrak{a} = (x^2 y, xy^2)$$

is not quasi-primary, because its radical, (xy) is not prime. $\mathfrak{a}$ has the primary representation

$$\mathfrak{a} = [(x), (y), (x^2, y^2)],$$

where the ideals (x) and (y) are prime, and (x^2, y^2) is primary belonging to the prime ideal (x, y).

From the radicals of the primary components of $\mathfrak{a}$ it is clear that both $[(x), (x^2, y^2)] = (x^2, xy^2)$ and $[(y), (x^2, y^2)] = (x^2 y, y^2)$ are quasi-primary; hence we have for $\mathfrak{a}$ the quasi-primary representations:

$$\mathfrak{a} = [(x), (x^2 y, y^2)] = [(x^2, xy^2), (y)] = [(x^2, xy^2), (x^2 y, y^2)]$$

with the prime ideals (x) and (y) respectively.

On the other hand, the process of NOETHER used above shows that $\mathfrak{a}$ is a relative-prime-irreducible ideal itself.

This example states that the quasi-primary representation of an ideal is, in general, different from the others and so, indeed, we have a right to consider the representation of an ideal by means of quasi-primary ideals as a fifth one.

§ 3. Applications to the theory of algebraic rings and polynomial ideals.

In the first part of this final section, we examine how our results change, if we suppose besides the maximal condition further axioms, especially[14]):

[14]) These axioms are those of B. L. VAN DER WAERDEN, loc. cit.[6]), p. 84. For the notion of entirely-closed ("ganz-abgeschlossen"), see ibidem, p. 78.

I. *every prime ideal is maximal in* $\mathfrak{R}$;

II. *the ring* $\mathfrak{R}$ *is entirely-closed in its quotient-field.*

In the first case the following theorem will be applied.

Theorem 7. *If the prime ideal* $\mathfrak{p}$ *belonging to the quasi-primary ideal* $\mathfrak{q}$ *is maximal in* $\mathfrak{R}$ *(i. e. has no proper divisor), then* $\mathfrak{q}$ *is primary.*

Let us consider the representation of $\mathfrak{q}$ as intersection of primary ideals:

$$\mathfrak{q} = [\mathfrak{q}_1^*, \ldots, \mathfrak{q}_t^*];$$

$\mathfrak{q}$ is quasi-primary, so its radical

$$\mathfrak{p} = [\mathfrak{p}_1, \ldots, \mathfrak{p}_t]$$

is prime ($\mathfrak{p}_\nu$ belongs to $\mathfrak{q}_\nu^*$). This is possible only in the case $\mathfrak{p}_1 = \ldots = \mathfrak{p}_t = \mathfrak{p}$, because $\mathfrak{p}$ has no proper divisor. Now $\mathfrak{q}$ is primary as the least common multiple of the primary ideals $\mathfrak{q}_\nu^*$ belonging to the same prime ideal $\mathfrak{p}$.[15])

Thus, if axiom I is true, every quasi-primary ideal is at the same time primary. Now, the intersection of the (quasi-) primary ideals belonging to different prime ideals is their product[16]), therefore, every ideal is the product of a finite number of (quasi-) primary ideals,

$$\mathfrak{a} = \mathfrak{q}_1 \ldots \mathfrak{q}_s,$$

and the prime ideals $\mathfrak{p}_i$ belonging to $\mathfrak{q}_i$ $(i = 1, \ldots, s)$ are uniquely determined.

In the case II the following theorem is obtained.

Theorem 8. *Every quasi-primary ideal is quasi-equal*[17]) *to* $\mathfrak{o}$ *or to a power of a prime ideal, if axiom II holds.*

A quasi-primary ideal has at most one *upper*[18]) prime ideal as divisor; so $\mathfrak{q}$ is quasi-equal to $\mathfrak{o}$ or to a product of equal upper prime ideals according as the prime ideal $\mathfrak{p}$ belonging to $\mathfrak{q}$ is not upper or it is. In the latter case the quasi-primary ideal is quasi-equal to a power of a prime ideal, q. e. d.

If in the ring $\mathfrak{R}$ the axioms I and II are simultaneously satisfied, then the principal theorem of the theory of ideals holds: *every ideal is the product of a finite number of uniquely determined prime-ideal-powers.*

Finally, we shall apply our results to ideals of a polynomial ring with coefficients in a commutative field[19]).

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[15]) See B. L. van der Waerden, loc. cit.[6]), p. 32.

[16]) See B. L. van der Waerden, loc. cit.[6]), p. 45.

[17]) See B. L. van der Waerden, loc. cit.[6]), p. 93.

[18]) See B. L. van der Waerden, loc. cit.[6]), p. 96; he calls it "höheres Ideal".

[19]) The basis-theorem of Hilbert (see e. g. B. L. van der Waerden, loc. cit.[6]), p. 18) states that in this polynomial ring the maximal condition is satisfied. — The rudiments of the theory of polynomial ideals see ibidem, § 91.

The algebraic manifold $\mathfrak{M}$ of the polynomial ideal $\mathfrak{m}$ consists of the zeros of the polynomials of $\mathfrak{m}$. Now, to find the irreducible manifolds belonging to $\mathfrak{M}$, let us take a shortest representation of $\mathfrak{m}$ as intersection of quasi-primary ideals:

$$\mathfrak{m} = [\mathfrak{q}_1, \ldots, \mathfrak{q}_r].$$

$\mathfrak{M}$ is the union of the manifolds of the quasi-primary components. Thus it suffices to examine the irreducible manifolds of a quasi-primary ideal.

Let us consider the quasi-primary ideal $\mathfrak{q}$ and its radical, the prime ideal $\mathfrak{p}$. $\mathfrak{q}$ and $\mathfrak{p}$ belong to the same manifold. Indeed, the zeros of $\mathfrak{q}$ are at the same time zeros of $\mathfrak{p}$ and vice versa, because a relation

$$\mathfrak{p}^n \subseteqq \mathfrak{q} \subseteqq \mathfrak{p}$$

holds. The manifold of a prime ideal is irreducible and so the manifold of a quasi-primary ideal is also irreducible. Moreover, an irreducible manifold can belong only to a quasi-primary ideal:

Theorem 9. *The algebraic manifold* $\mathfrak{N}$ *of the ideal* $\mathfrak{a}$ *is irreducible if and only if* $\mathfrak{a}$ *is quasi-primary.*

The first part of our theorem we have already proved. To prove the second part of it, consider the radical $\mathfrak{r}$ of $\mathfrak{a}$. Even the ideal $\mathfrak{r}$ belongs to $\mathfrak{N}$, because the relation

$$\mathfrak{r}^n \subseteqq \mathfrak{a} \subseteqq \mathfrak{r}$$

is true for any ideal $\mathfrak{a}$ and for its radical $\mathfrak{r}$. (The proof runs as that of theorem 2.) We have seen that $\mathfrak{r}$ is either prime or the intersection of a finite number of prime ideals,

$$\mathfrak{r} = [\mathfrak{p}_1, \ldots, \mathfrak{p}_s].$$

(We may suppose that here the superfluous prime ideals have been already omitted.) In the latter case, we have the manifold $\mathfrak{N}$ consisting of the manifolds belonging to $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$, that is to say, $\mathfrak{N}$ is not irreducible. This is a contradiction to the hypothesis. Thus $\mathfrak{r}$ is prime and finally, $\mathfrak{a}$ is quasi-primary. This completes the proof of theorem 9.

Returning to the ideal $\mathfrak{m}$ we get the result:

Theorem 10. *The algebraic manifold of an ideal* $\mathfrak{m}$ *is the union of a finite number of irreducible algebraic manifolds. These irreducible algebraic manifolds are just all the irreducible manifolds belonging to the quasi-primary ideals of a shortest representation of* $\mathfrak{m}$.

Therefore, the representation of an ideal as intersection of quasi-primary ideals can be considered as the natural representation of the ideal from the point of view of algebraic manifolds.

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