A series associated with Dirichlet's series.

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The object of this note is to make explicit the fact that we can associate with the Dirichlet series

$$\text{(D)} \qquad \sum_{1}^{\infty} a_n/D_n^s, \qquad 0=D_0<D_1<\ldots, D_n\to\infty,$$

whenever $d_n \equiv D_n - D_{n-1} = O(1)$,[1] a generalised factorial series

$$\text{(F)} \qquad \sum_{1}^{\infty} \frac{a_n}{\left(1+\frac{d_1}{D_1}s\right)\left(1+\frac{d_2}{D_2}s\right)\ldots\left(1+\frac{d_n}{D_n}s\right)},$$

the association implying that (F) reflects some of the well-known properties of (D) including those familiar to us as Tauberian theorems. It will be recalled in this connection that LANDAU [5] has considered a series more general than (F), with $\{d_n/D_n\}$ changed to a positive sequence $\{p_n\}$ satisfying the conditions: $p_n\to 0$, $p_1+p_2+\ldots+p_n\to\infty$. A Tauberian theorem which he proves for such a series yields Theorem II below, however, with o in place of O in the condition imposed on a_n.

Theorem I. (F) *converges — with the exclusion of the points* $s=-D_n/d_n$ $(n=1,2,3,\ldots)$ *— whenever* (D) *converges; and conversely,* (D) *converges whenever* (F) *does so. The convergence is uniform in any finite region of the s-plane for either series, when it is so for the other, provided the region contains none of the points* $-D_n/d_n$.

Proof. We have only to note that

the nth term of (D) $=$ the nth term of (F) $\times P(s, d_n)$,

where

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[1] This assumption, supposed to hold throughout our discussion, makes $\Sigma(d_n/D_n)^{\mu}$ convergent when $\mu>1$ and divergent when $\mu=1$; in the latter case ensuring a finite value for $\lim\left(\frac{d_1}{D_1}+\ldots+\frac{d_n}{D_n}-\log D_n\right)$ as $n\to\infty$. It is further supposed that a_n and s are complex whenever their reality is not definitely affirmed.

$$(1) \qquad P(s, d_n) = \left(1+\frac{d_1}{D_1}s\right)\ldots\left(1+\frac{d_n}{D_n}s\right)\Big/ D_n^s$$

and appeal to Abel's test for (uniform) convergence, justifying this appeal by

Lemma 1. *The series*

$$(2) \qquad \sum\left|P(s,d_n)-P(s,d_{n-1})\right|, \quad \sum\left|\frac{1}{P(s,d_n)}-\frac{1}{P(s,d_{n-1})}\right|$$

converge uniformly in any finite part of the s-plane from which, in the case of the latter series, the points $-D_n/d_n$ $(n=1,2,3,\ldots)$ *are excluded.*

Proof. For $n \geqq 2$,

$$P(s,d_n)-P(s,d_{n-1}) = P(s,d_{n-1})\left[\left(1+\frac{d_n}{D_n}s\right)\left(1-\frac{d_n}{D_n}\right)^s-1\right]$$
$$= P(s,d_{n-1})\left[-\frac{d_n^2}{D_n^2}s^2+\frac{d_n^2}{D_n^2}\vartheta_n(s)+\frac{d_n^3}{D_n^3}s\vartheta_n(s)\right]$$

where it is easy to prove that $|P(s,d_{n-1})|<A, |\vartheta_n(s)|<B$, A and B being constants independent of s when s is restricted to a finite region. This proves that $|P(s,d_n)-P(s,d_{n-1})|<K d_n^2/D_n^2$ and so establishes the conclusion in respect of the first series in (2). The conclusion in respect of the second series in (2) can be proved similarly since, in any finite s-region from which we have excluded $s=-D_n/d_n$ $(n=1,2,\ldots)$, we have $1/|P(s,d_{n-1})|<A'$ independent of s.

The details which I have omitted from the foregoing proofs are supplied by the similar discussion of the case $d_n \equiv 1$ which is well known [3, pp. 440f., 446f.].

Theorem II. *If* (F) *converges for* $\mathrm{Rl}\, s>0$, *its sum being denoted by* $F(s)$, *and if* (*i*) $F(s)\to C$ *as* $s\to 0$ *through real positive values,* (*ii*) $a_n=O[d_n/D_n \log D_n]$,[2]) *then* $\Sigma a_n = C$.

This theorem is an immediate consequence of the next two lemmas of which the first, in a slightly different form, is a well-known result of HARDY and LITTLEWOOD [2, Theorem F].

Lemma 2. *If* (D) *converges for* $\mathrm{Rl}\, s>0$ *to the sum* $D(s)$ *and if* (*i*) $D(s)\to C$ *as* $s\to 0$ *through positive values,* (*ii*) $a_n=O[d_n/D_n \log D_n]$, *then* $\Sigma a_n = C$.

Lemma 3. *If* $a_n=o(d_n/D_n)$, *then* $F(s)-D(s)\to 0$ *as* $s\to 0$ *through positive values; and so, in particular, when* $a_n=O[d_n/D_n \log D_n]$,

$$F(s)\to C \text{ implies } D(s)\to C.$$

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[2]) This condition implies, when $\mathrm{Rl}\, s>0$, the (absolute) convergence of (D) and hence, by Theorem I, the convergence of (F).

Proof. Let $s>0$ be chosen so that $sd_n/D_n<1$ for $n=1, 2, \ldots$ Then, if $P(s, d_n)$ is defined by (1), we have

$$\exp\left[\sum_1^3 c_r(n)s^r\right] < \frac{1}{P(s, d_n)} < \exp\left[\sum_1^2 c_r(n)s^r\right]$$

where

$$c_1(n) = \log D_n - \frac{d_1}{D_1} - \ldots - \frac{d_n}{D_n},$$

$$c_r(n) = \frac{(-1)^r}{r}\left[\frac{d_1^r}{D_1^r} + \ldots + \frac{d_n^r}{D_n^r}\right], \quad r=2, 3, \ldots,$$

tend to finite limits as $n\to\infty$. Consequently

$$\frac{1}{P(s, d_n)} = e^{O(s)}, \quad s\to+0, \tag{3}$$

uniformly with respect to n.

Now

$$D(s)-F(s) = \sum_1^N \frac{a_n}{D_n^s}\left[1-\frac{1}{P(s, d_n)}\right] + \sum_{N+1}^{\infty} \frac{a_n}{D_n^s}\left[1-\frac{1}{P(s, d_n)}\right] \tag{4}$$

where (as a result of the condition satisfied by a_n) N can be chosen so that, for $n>N$,

$$|a_n| < \varepsilon d_n/D_n$$

and therefore

$$\left|\sum_{N+1}^{\infty}\ldots\right| < \varepsilon \sum_{N+1}^{\infty} \frac{d_n}{D_n^{s+1}}\left|1-\frac{1}{P(s, d_n)}\right|. \tag{5}$$

Using (3) in each term on the right-hand side of (5) we get

$$\left|\sum_{N+1}^{\infty}\ldots\right| < \varepsilon K s \sum_{N+1}^{\infty} \frac{d_n}{D_n^{s+1}}, \qquad 0<s<s_0,$$

and thence, recalling that

$$\lim_{s\to+0} s\sum_{N+1}^{\infty}\frac{d_n}{D_n^{s+1}} = 1 \quad \text{when} \quad \lim_{n\to\infty}\frac{D_{n+1}}{D_n} = 1,$$[3])

we are led to

$$\left|\sum_{N+1}^{\infty}\ldots\right| \to 0, \quad s\to+0. \tag{6}$$

Employing (6) in (4), where obviously $\left|\sum_1^n\ldots\right|\to 0$ as $s\to+0$, we complete the proof of the lemma.

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[3]) For a straightforward proof of results of this type, the reader is referred to my discussion cited at the end [6].

An obvious modification of Lemma 3 is

Lemma 3a. *If* $a_n = O(d_n/D_n)$, *then* $F(s) - D(s) = O(1)$ *as* $s \to 0$ *through positive values.*

Remark. The Abelian theorem which is the converse of Theorem II follows from the uniform convergence of (D) reflected by (F) as stated in Theorem I. It can be enunciated thus:

If $\Sigma a_n = C$, *then* $F(s)$, *representing the sum of* (F) *for* $\mathrm{Rl}\, s > 0$, *tends to* C *as* $s \to 0$ *along any path lying entirely within the region* $|\mathrm{am}\ s| \leqq \varkappa < \pi/2$.

Theorem III. *Let* $a_n \geqq 0$. *Let* (F) *be convergent for* $s > 0$, *its sum* $F(s)$ *satisfying the condition*

$$F(s) \sim C s^{-\alpha} (l_1 s^{-1})^{\alpha_1} \dots (l_m s^{-1})^{\alpha_m}, \quad s \to +0, \tag{7}$$

where $l_r u = \log\log\dots$ *(r times)* u, $\alpha \geqq 0$ *and either the first non-zero* α *is positive or every* α *is zero. Then*

$$a_1 + a_2 + \dots + a_n \sim C \frac{(l_1 D_n)^{\alpha} (l_2 D_n)^{\alpha_1} \dots (l_{m+1} D_n)^{\alpha_m}}{\Gamma(\alpha + 1)}. \tag{8}$$

Conversely (7) *follows from* (8) *when* $a_n \geqq 0$.

This theorem is obtained by combining the two lemmas given below.

Lemma 4. *Let* $a_n \geqq 0$. *Then* (7) *implies and is implied by the similar relation in which* $F(s)$ *is replaced by* $D(s)$, *the sum of* (D) *for* $s > 0$.

Proof. For $s > 0$, let us write

$$D_N(s) = \sum_{N+1}^{\infty} \frac{a_n}{D_n^s}.$$

$$F_N(s) = \sum_{N+1}^{\infty} \frac{a_n}{\left(1 + \frac{d_1}{D_1} s\right)\left(1 + \frac{d_2}{D_2} s\right) \dots \left(1 + \frac{d_n}{D_n} s\right)} = \sum_{N+1}^{\infty} \frac{a_n}{D_n^s} \frac{1}{P(s, d_n)},$$

so that $D_0(s) = D(s)$, $F_0(s) = F(s)$. Let

$$\frac{1}{\Pi_d(s)} = \lim_{n \to \infty} P(s, d_n) = e^{\nu_d s} \prod_{n=1}^{\infty} \left(1 + \frac{d_n}{D_n} s\right) e^{-s d_n / D_n},$$

where $\nu_d = \lim_{n \to \infty} \left(\frac{d_1}{D_1} + \frac{d_2}{D_2} + \dots + \frac{d_n}{D_n} - \log D_n \right)$, denote the reciprocal of CESÀRO's generalization of GAUSS's function $\Pi(s) = \Gamma(s + 1)$.

Then we can choose N so that, for $n > N$,

$$\frac{1 - \varepsilon}{\Pi_d(s)} < P(s, d_n) < \frac{1 + \varepsilon}{\Pi_d(s)}$$

and hence

$$\frac{1 - \varepsilon}{\Pi_d(s)} F_N(s) < D_N(s) < \frac{1 + \varepsilon}{\Pi_d(s)} F_N(s). \tag{9}$$

Since (7) implies

$$F_N(s) \sim Cs^{-\alpha}(l_1 s^{-1})^{\alpha_1} \dots (l_m s^{-1})^{\alpha_m}, \quad s \to +0,$$

and we have also $\Pi_d(s) \to 1$ as $s \to +0$, it follows from (9) that

$$D_N(s) \sim Cs^{-\alpha}(l_1 s^{-1})^{\alpha_1} \dots (l_m s^{-1})^{\alpha_m}, \quad s \to +0$$

and therefore

$$D(s) \sim Cs^{-\alpha}(l_1 s^{-1})^{\alpha_1} \dots (l_m s^{-1})^{\alpha_m}, \quad s \to +0. \tag{10}$$

Thus, when $a_n \geqq 0$, (7) involves (10) and similarly (10) involves (7)[4].

Lemma 5. *Let $a_n \geqq 0$; then* (10) *involves* (8). *Conversely* (8) *involves* (10), *even without any explicit assumption regarding a_n.*

The first part of the lemma is a classical result of HARDY and LITTLEWOOD [2, Theorem D and p. 141, footnote †]; the second (converse) part can be proved by a method suggested by GANAPATHY IYER [1, § 7].

Theorem IV. *Let a_n be real and satisfy the condition*

$$a_n \geqq -K \frac{d_n}{D_n (l_1 D_n)^{1-\alpha}}, \quad 0 < \alpha \leqq 1.$$

Let (F) *be convergent for $s > 0$, its sum $F(s)$ being such that*

$$F(s) \sim Cs^{-\alpha}, \quad s \to +0.$$

Then

$$a_1 + a_2 + \dots + a_n \sim \frac{C(l_1 D_n)^\alpha}{\Gamma(\alpha+1)}.$$

Proof. The series $\Delta(s) = \Sigma d_n / D_n^{s+1}(l_1 D_n)^{1-\alpha}$ is convergent for $s > 0$ and hence, by Theorem I, also

$$\Phi(s) = \frac{d_n}{D_n (l_1 D_n)^{1-\alpha}} \Bigg/ \left(1 + \frac{d_1}{D_1} s\right) \dots \left(1 + \frac{d_n}{D_n} s\right).$$

Since $\Delta(s) \sim \Gamma(\alpha) s^{-\alpha}$, $\alpha > 0$, $s \to +0$,[5]) we can take $a_n = d_n / D_n (l_1 D_n)^{1-\alpha}$ in Lemma 3a and obtain

$$\Phi(s) \sim \Gamma(\alpha) s^{-\alpha}, \quad 0 < \alpha \leqq 1, \quad s \to +0.$$

The series

$$G(s) = F(s) + K\Phi(s)$$

is got by changing a_n in (F) to

$$b_n = a_n + K d_n / D_n (l_1 D_n)^{1-\alpha} \geqq 0$$

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[4]) The argument tacitly assumes that $s^{-\alpha}(l s^{-1})^{\alpha_1}, \dots (l_m s^{-1})^{\alpha_m} \to \infty$ as $s \to +0$. The case in which $\alpha = \alpha_1 = \dots = \alpha_m = 0$ is dealt with by an obvious modification of the argument.

[5]) Knopp's proof of this result for $d_n \equiv 1$ [4, pp. 180–1] can obviously be modified when $d_n = O(1)$ as we have supposed.

and is subject to the condition

$$G(s) \sim \{C + K\Gamma(\alpha)\}\, s^{-\alpha}, \quad s \to +0.$$

Hence, by Theorem III,

$$\sum_{1}^{n} b_\nu \sim \left\{ \frac{C}{\Gamma(\alpha+1)} + \frac{K}{\alpha} \right\} (l_1 D_n)^\alpha$$

and since, when $\alpha > 0$ and $\lim\limits_{n \to \infty} D_{n+1}/D_n = 1$,

$$\sum_{1}^{n} \frac{d_\nu}{D_\nu (l_1 D_2)^{1-\alpha}} \sim \frac{(l_1 D_n)^\alpha}{\alpha},$$

it follows that

$$\sum_{1}^{n} a_\nu \sim \frac{C (l_1 D_n)^\alpha}{\Gamma(\alpha+1)}.$$

It may be mentioned in conclusion that Theorems I—IV can be restated with the series

$$\text{(F')} \qquad \sum_{1}^{\infty} a_n \left(1 - \frac{d_1}{D_1} s\right) \dots \left(1 - \frac{d_n}{D_n} s\right)$$

taking the place of the series (F) and with the obvious difference that in Theorem I points $(s = D_n/d_n)$ are excluded only when the convergence of (F') is posited.

References.

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