On ellipsoids circumscribed and inscribed to polyhedra.

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1. The ellipses (ellipsoids) circumscribed and inscribed to a polygon (polyhedron) Π are defined as the ellipse (ellipsoid) of smallest area (volume) containing Π, and that of largest area (volume) contained in Π, respectively. We shall denote in this paper a domain and its area (volume) by the same letter.

As a generalization of the fact that the radius of the circumscribed circle of an arbitrary triangle is at least twice as large as the radius of the inscribed circle[1]), we have the following proposition[2]):

If E_n and e_n are the circumscribed and inscribed ellipses of an n-sided polygon, then

$$E_n/e_n \geqq \cos^{-2}\frac{\pi}{n}.$$

Equality holds only for the affine images of regular polygons. Hence the ellipses circumscribed and inscribed to the "best" n-sided polygon — i. e. for which E_n/e_n takes its minimal value — are concentric and homothetic.

The analogous question in the space is to find among the polyhedra with n vertices or n faces those which minimize the quotient E_n/e_n of the volumes of the circumscribed and inscribed ellipsoids.

It follows from the nature of the problem that we cannot expect to obtain in such a simple manner the set of the best polyhedra for all values of n as in the two-dimensional case. Thus the question arises

1) To this fact and to the analogous problem for tetrahedra my attention was turned by Professor L. FEJÉR, who remarked the above inequality in 1897 as a competitor at the mathematical competition of the Loránd Eötvös Mathematical and Physical Society. *Cf:* J. KÜRSCHÁK, *Matematikai versenytételek* (Szeged, 1929); T. RADÓ, On mathematical life in Hungary, *American Math. Monthly,* 39 (1932), pp. 85—90.

2) *Cf.* L. FEJES TÓTH, An inequality concerning polyhedra, *Bulletin of the American Math. Society* (in the press), where — in footnote 5) — the affirmative part of the theorem below is also announced. We return to this question in view of the result in the negative direction.

whether at least — analogously to the two-dimensional problem — the ellipsoids circumscribed and inscribed to the best n-verticed or n-faced polyhedron are for all values of n concentric and homothetic, or they are not.

The answer to this question is given by the following

Theorem. *Consider the set of the pairs* $\{E_n, e_n\}$ *of ellipsoids circumscribed and inscribed to convex polyhedra, having either a given number n of vertices or a given number n of faces. In both cases the ellipsoids of any pair for which* E_n/e_n *takes its minimal value, are concentric but generally not homothetic.*

The natural and apparently easier question whether for the n-verticed or n-faced polyhedra which minimize the quotient R_n/r_n of the radii of the spheres containing and contained in the polyhedron, the corresponding spheres are concentric for all $n \geqq 4$ or not, is still undecided.

2. The affirmative part of the theorem announced above is a consequence of the following

Lemma. *If* E_1, E_2, E_3 *denote three ellipsoids,* E_1 *and* E_3 *being polar reciprocals of each other with respect to* E_2*, then*

$$E_1/E_2 \geqq E_2/E_3.$$

Equality holds only if E_1, E_2, E_3 *are concentric.*

It may be supposed that E_2 is the unit sphere with its centre at the origin, and that the x, y, z-axes are parallel to the principal axes $2a, 2b, 2c$ of E_1, respectively. Let ξ, η, ζ be the coordinates of the centre of E_1. Since, by hypothesis, the ellipsoid E_1 is carried by the polar reciprocity with respect to E_2 into an ellipsoid, it follows that there is no tangent plane of E_1 passing through the centre of E_2. (For to such a plane would correspond, by the polar reciprocity, a point at the infinity.) Consequently, E_1 contains the centre of E_2, thus $|\xi| < a, |\eta| < b, |\zeta| < c$.

The reciprocity, applied to the tangent planes of E_1 at the end-points of the axis of length $2a$, yields two points of E_3 lying on the x-axis, the distance of which is given by

$$\frac{1}{a+\xi}+\frac{1}{a-\xi}=\frac{2a}{a^2-\xi^2}\geqq\frac{2}{a}.$$

Similar considerations applied to the other axes of E_1 yield three mutually perpendicular chords of E_3 whose lengths are not less than $2/a$, $2/b$, $2/c$, respectively.

The diameters 2α, 2β, 2γ of E_3 parallel to these chords are, *a fortiori*, $\geqq 2/a$, $2/b$, $2/c$, respectively.

Consider the octahedron Ω with diameters $2\alpha, 2\beta, 2\gamma$. We have $\Omega = \frac{4}{3}\alpha\beta\gamma \geqq \frac{4}{3}\frac{1}{abc}$.

Let us replace 2α by the diameter $2\alpha'$ of E_3 conjugate with respect to E_3 to the diametral plane $\beta\gamma$. Similarly, let us replace 2β by the diameter $2\beta'$ conjugate to the diametral plane $\alpha'\gamma$. The volume of the octahedron Ω has been increased by both steps. The diameters α', β', γ of the new octahedron Ω' are pair by pair conjugate with respect to E_3 and thus $\Omega' = \frac{4}{3}\bar{\alpha}\bar{\beta}\bar{\gamma}$, where $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ denote the principal axes of E_3. Since $\Omega' \geqq \Omega$, we have $E_3 = \frac{4\pi}{3}\bar{\alpha}\bar{\beta}\bar{\gamma} \geqq \frac{4\pi}{3}\frac{1}{abc}$, i. e.

$$E_1 E_3 \geqq \frac{4\pi}{3} abc \frac{4\pi}{3} \frac{1}{abc} = \left(\frac{4\pi}{3}\right)^2 = E_2^2,$$

which proves the lemma. Equality holds only if $\xi = \eta = \zeta = 0$. In this case E_3 is also concentric with E_2 and its principal axes are $2/a$, $2/b$, $2/c$.

Let us suppose now that the ellipsoids E_n and e_n circumscribed and inscribed to the best n-verticed (n-faced) polyhedron P_n are not concentric. Taking polar reciprocals with respect to e_n, there corresponds to P_n a polyhedron P_n' with n-faces (vertices) contained in e_n and containing the ellipsoid E_n' reciprocal to E_n. A second polar reciprocity with respect to E_n' carries P_n' into a polyhedron P_n'' with n vertices (faces) contained in E_n' and containing the ellipsoid e_n' reciprocal to e_n. According to the lemma and to our hypothesis, we have

$$E_n/e_n > e_n/E_n' > E_n'/e_n'.$$

Hence P_n'' is better than P_n; this contradiction proves the theorem.

Since the ellipsoids of any extremal pair are generally not homothetic, the result proved just now is the most which can be said in this direction and it is surprising that this result could be obtained by such simple means, without using any of the more intricate properties of polyhedra.

3. Let us now turn to the negative part of our theorem.

The above considerations show that the minimal value of E_n/e_n for polyhedra having n vertices is equal to the minimal value of E_n/e_n for polyhedra having n faces and the best n-verticed and n-faced polyhedra are mutually polar reciprocals of each other with respect to the inscribed or circumscribed ellipsoid.

Therefore we can restrict ourselves to n-verticed polyhedra.

The circumscribed and inscribed ellipsoids of a tetrahedron are — as affine images of two concentric spheres — always homothetic. But the case $n = 5$ furnishes already the required exemple of an extremal pair E_5, e_5 of ellipsoids which are not homothetic.

A convex polyhedron with 5 vertices is generally a 6-faced double pyramid which can degenerate to a 4-sided pyramid (or to the convex envelope of 4 or less points).

It follows immediately that among the 5-verticed polyhedra P_5 contained in a sphere S, the double pyramid d, formed by the vertices of an equilateral triangle inscribed in the equator and of the two poles, has the greatest volume.

Somewhat more complicated is to determine the 5-verticed polyhedron D containing the sphere S, which has the least volume.

LHUILIER[3]) determined the $2m$-faced double pyramid D_m having the minimal value of $|D_m|^3/D_m^2$ where the sign of the absolute value denotes the surface area. D_m is composed by two congruent straight pyramids with regular m-goned bases, so that the greatest sphere S contained in D_m touches all faces at their centre of gravity.

We assert that D_3 is at the same time the polyhedron D having the least volume among the polyhedra P_5 with 5 vertices containing the sphere S of radius r and centre O. For decompose P_5 into 6 tetrahedra having the common vertex O. The altitudes of these tetrahedra being $\geqq r$, we have $P_5 \geqq \frac{r}{3}|P_5|$. Hence indeed, we have for all P_5 incongruent to D_3:

$$P_5 \geqq \frac{r^3}{27}\frac{|P_5|^3}{P_5^2} > \frac{r^3}{27}\frac{|D_3|^3}{D_3^2} = D_3.$$

From the extremum properties of the polyhedra d and D we obtain by well known properties of the affinity, for all 5-verticed convex polyhedra P_5:

$$e_5\frac{D}{S} \leqq P_5 \leqq E_5\frac{d}{S}$$

and hence

$$E_5/e_5 \geqq D/d.$$

The lower bound on the right side is reached e. g. for the 5-verticed double pyramid d. In this case E_5 is a sphere; on the other hand e_5 is an ellipsoid of revolution which touches the faces of d in their centre of gravity.

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[3]) S. LHUILIER, *De relatione mutua capacitatis et terminorum figurarum, etc.* (Varsaviae, 1782).