On the structure of groups which can be represented as the product of two subgroups.

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The results of the present paper given here are connected with the researches of L. RÉDEI[1]) on the generalization of the skew product *(schiefes Produkt)* introduced by him. I shall return to this connection in the detailed discussion. For a long time I believed that my researches are connected only with those of L. RÉDEI. Shortly before publication I learned that G. ZAPPA[2]) has arrived in 1940 at the relations (2), (3), (4) of the present paper in a similar way as I did.

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In a previous paper[3]) I have published the following results:

Let the group $\mathfrak{G}$ be representable as the product of two proper subgroups $\mathfrak{H}$, $\mathfrak{K}$

(a) $$\mathfrak{G}=\mathfrak{H}\mathfrak{K}.$$

This means that each element of $\mathfrak{K}$ multiplied by each element of $\mathfrak{H}$ produces all elements of $\mathfrak{G}$ exactly once. It is well known that the necessary and sufficient condition for the fulfilment of (a) is, that the only common element of the subgroups $\mathfrak{H}$ and $\mathfrak{K}$ should be the unit element. For the order of the group $\mathfrak{G}$ the equality $(\mathfrak{G})=(\mathfrak{H})\cdot(\mathfrak{K})$ holds. Owing to the symmetrical position of $\mathfrak{H}$ and $\mathfrak{K}$, (a) implies

(b) $$\mathfrak{K}\mathfrak{H}=\mathfrak{H}\mathfrak{K}.$$

Let $H, H', \bar{H}, \ldots$ and $K, K', \bar{K}, \ldots$ denote elements of $\mathfrak{H}$ and $\mathfrak{K}$, respectively. Then by (b) we get a relation

(c) $$KH=H'K'.$$

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[1]) L. RÉDEI, Die Anwendungen des schiefen Produktes in der Gruppentheorie, *Journal für die reine und angewandte Math.* (Under press.) In this paper, RÉDEI defines a group $\mathfrak{G}$ arising from two given groups $\mathfrak{H}$ and $\mathfrak{K}$ by relations which are essentially identical with the relations (2), (3), (4). He showed that this group $\mathfrak{G}$ has subgroups isomorphic to $\mathfrak{H}$ and $\mathfrak{K}$, and that their orders satisfy the relation $(\mathfrak{G})=(\mathfrak{H})(\mathfrak{K})$. He gives a new proof, and partly an improvement, of our results contained in §§ 1—2.

[2]) G. ZAPPA, Costituzione dei gruppi prodotte di due dati sottogruppi permutabili tra loro, *Atti Secondo Congresso Unione Mat. Italiana Bologna* 1940, p. 115—125.

[3]) J. SZÉP, Über die als Produkt zweier Untergruppen darstellbaren endlichen Gruppen, *Commentarii Math. Helvetici,* **22** (1948), p. 31–33.

where H' and K' are uniquely defined whenever both H and K are given. When H fixed, then K' together with K runs over all elements of $\mathfrak{K}$. We may thus associate with each H the following permutation of elements of $\mathfrak{K}$

$$H \longrightarrow \Pi_H = \begin{pmatrix} K \\ K' \end{pmatrix}.$$

It is readily seen that $\Pi_H \Pi_{\bar{H}} = \Pi_{H\bar{H}}$ holds.

Theorem A: *The permutations Π_H of the elements of $\mathfrak{K}$ form a group $\Pi(\mathfrak{K})$ and $\mathfrak{H} \sim \Pi(\mathfrak{K})$. In this homomorphism to the unit element of $\Pi(\mathfrak{K})$ corresponds a maximal normal subgroup $\mathfrak{N}$ of $\mathfrak{G}$[4]) which is a normal subgroup of $\mathfrak{H}$ and*

$$\mathfrak{H}/\mathfrak{N} \cong \Pi(\mathfrak{K}).$$

In an exactly similar way, leaving K fixed in $H'K' = KH$, H' runs together with H over all elements of $\mathfrak{H}$. Again, we associate with the element K the permutation $\Pi_K = \begin{pmatrix} H \\ H' \end{pmatrix}$ of the elements of $\mathfrak{H}$. As easily seen, $\Pi_{\bar{K}} \Pi_K = \Pi_{K\bar{K}}$.

Owing to the symmetry of $\mathfrak{H}$ and $\mathfrak{K}$, theorem A holds also if we change their rôle.

§ 1.

We shall introduce a new notation for the permutations Π and for H' and K' in (c):

$$KH = H^{[K]} K^{[H]}, \tag{1}$$

that is, we denote by $K^{[H]}$ the element K' into which K passes by the permutation $[H] = \Pi_H$. The notation $H^{[K]}$ should be understood in a similar way.

By (1), the product of two elements of $\mathfrak{G}$ may be written as

$$HKH'K' = HH'^{[K]} K^{[H']} K'. \tag{2}$$

The elements of $\mathfrak{G}$ have to satisfy the associative law

$$(HKH'K')H''K'' = HH'^{[K]} K^{[H']} K' H'' K'' = HH'^{[K]} H''^{[K^{[H']}K']} (K^{[H']} K')^{[H'']} K'',$$

$$HK(H'K'H''K'') = HKH'H''^{[K']} K'^{[H'']} K'' = H(H'H''^{[K']})^{[K]} K^{[H'H''^{[K']}]} K'^{[H'']} K''.$$

Comparing these we get

$$(H'H''^{[K']})^{[K]} = H'^{[K]} H''^{[K^{[H']}K']}, \quad (K^{[H']} K')^{[H'']} = K^{[H'H''^{[K']}]} K'^{[H'']}.$$

As these relations must hold for every H and K, they may be written in a similar form:

$$(HH')^{[K]} = H^{[K]} H'^{[K^{[H]}]}, \quad (KK')^{[H]} = K^{[H^{[K']}]} K'^{[H]}. \tag{3}$$

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[4]) This means that $\mathfrak{G}$ has no normal subgroup $\mathfrak{N}'$ $((\mathfrak{N}') > (\mathfrak{N}))$ which is a subgroup of $\mathfrak{H}$.

We may add to these the relations given by the definition of $H^{[K]}$ and $K^{[H]}$:

(4) $$(H^{[K]})^{[K']} = H^{[K][K']}, \quad (K^{[H]})^{[H']} = K^{[H][H']}.$$

It may easily be proved, that if we have for example $\mathfrak{H} \simeq [\mathfrak{K}]$ by the given permutations of $[\mathfrak{K}]$, the permutations of $[\mathfrak{H}]$ are uniquely defined.

§ 2.

The elements $[H]$ of the group $[\mathfrak{K}]$ are permutations of the group $\mathfrak{K}$. From now on let these permutations be automorphisms of the group $\mathfrak{K}$. If similar condition holds for the $[\mathfrak{H}]$ too, i. e., the elements $[K]$ of the group $[\mathfrak{H}]$ are automorphisms of the group $\mathfrak{H}$, we call the group $\mathfrak{G} = \mathfrak{H}\mathfrak{K}$ a group of automorphic composition. If only one of these two conditions holds, I shall denote the group $\mathfrak{G}$ a group of semiautomorphic composition.

Theorem 1. *Every group of automorphic or semiautomorphic composition has a proper normal subgroup.*

Proof. By (c) we have $KH = H'K'$. If $H' = H$ for every H, when K' runs over all elements of $\mathfrak{K}$ simultaneously with K, our theorem is evident, since $\mathfrak{K}H = H\mathfrak{K}$ holds for every H, i. e. $\mathfrak{K}$ is a normal subgroup.

Let now $H' \neq H$, then we have $KH = H'K'$, $\bar{K}H = \bar{H}'\bar{K}'$, thus we get $\bar{K}K^{-1}H' = \bar{H}'\bar{K}'K'^{-1}$. Since the permutation $[H] = \begin{pmatrix} K \\ K' \end{pmatrix}$ is an automorphism of the group $\mathfrak{K}$, we have in $[H]$: $\bar{K}K^{-1} \to \bar{K}'K'^{-1}$ and we get

$$[H'] = \begin{pmatrix} \bar{K}K^{-1} \\ \bar{K}'K'^{-1} \end{pmatrix} = \begin{pmatrix} K \\ K' \end{pmatrix} = [H],$$

thus $[H'][H]^{-1} = [H'H^{-1}] = E$. Consequently, according to theorem A, $\mathfrak{H}$ has a proper normal subgroup $\mathfrak{N}$, which is also a normal subgroup of $\mathfrak{G}$.

The proof runs similarly for $\mathfrak{K}$.

Corollary 1. 1. *Let $\mathfrak{H}$ and $\mathfrak{K}$ be given. If the order of the automorphism group of $\mathfrak{K}$ is relatively prime to the order of $\mathfrak{H}$ and the same holds for the automorphism group of $\mathfrak{H}$, then there is only one group $\mathfrak{G} = \mathfrak{H}\mathfrak{K}$ of automorphic composition and $\mathfrak{G}$ is the direct product of $\mathfrak{H}$ and $\mathfrak{K}$.*

Corollary 1. 2. *If the group $\mathfrak{G}$ is a group of automorphic composition and if $[\mathfrak{H}] \simeq \mathfrak{K}$, then $\mathfrak{H}$ is a normal subgroup.*

According to (1), $K \to K^{[H]}$, $K' \to K'^{[H]}$, ... In a group of automorphic composition we have $KK' \to (KK')^{[H]} = K^{[H]}K'^{[H]}$. Similarly, for the elements of $\mathfrak{H}$ we have $HH' \to (HH')^{[K]} = H^{[K]}H'^{[K]}$.

For the group $\mathfrak{G}=\mathfrak{H}\mathfrak{K}$ of automorphic composition the relations (2), (3), (4) become

(2') $$HKH'K'=HH'^{[K]}K^{[H']}K',$$

(3') $$(HH')^{[K]}=H^{[K]}H'^{[K]},\ (KK')^{[H]}=K^{[H]}K'^{[H]},$$

(4') $$(H^{[K]})^{[K']}=H^{[K][K']},\ (K^{[H]})^{[H']}=K^{[H][H']}.$$

Comparing (3') with (3),

(5) $$[K^{[H]}]=[K],\ [H^{[K']}]=[H];$$

(5) holds for every $H\in\mathfrak{H}$ and $K'\in\mathfrak{K}$. Let now $\mathfrak{N}\subset\mathfrak{H}$ be the maximal normal subgroup of $\mathfrak{G}$[4]) (defined by theorem A), further let $\mathfrak{M}\subset\mathfrak{K}$ similarly defined. Then (5) means that the automorphism $[H]$ transforms the element K into the coset of the factor-group $\mathfrak{K}/\mathfrak{M}$ containing K. Similar statement holds for $\mathfrak{H}/\mathfrak{N}$.

Hence in groups of automorphic composition, the elements of $[\mathfrak{K}]$ transform every coset $K\mathfrak{M}=F_K$ of the factor-group $\mathfrak{K}/\mathfrak{M}$ into itself, i. e., using the notation of (c), $F_K H=H'F_K$. Similarly, $[\mathfrak{H}]$ transforms every coset $H\mathfrak{N}=F_H$ of the factor-group $\mathfrak{H}/\mathfrak{N}$ into itself, $F_H K'=KF_H$. As $H'F_K=F_K H$, if H varies over all the elements of $\mathfrak{H}$, H' does the same; $\begin{pmatrix}H\\H'\end{pmatrix}=[K]=[F_K]$. The permutation $\begin{pmatrix}H\\H'\end{pmatrix}$ transforms F_H into itself, thus $F_H F_K=F_K F_H$ for every $H\in\mathfrak{H}$ and $K\in\mathfrak{K}$.

Since $\mathfrak{M}$ and $\mathfrak{N}$ have no common element other than the unit element, $\mathfrak{M}\times\mathfrak{N}$ is again a normal subgroup of $\mathfrak{G}$ (the sign $\times$ denotes direct product). Hence we get the following theorem:

T h e o r e m 2. *Let $\mathfrak{G}=\mathfrak{H}\mathfrak{K}$ be a group of automorphic composition, let $\mathfrak{N}\subset\mathfrak{H}$ be the maximal normal subgroup of $\mathfrak{G}$ and $\mathfrak{M}$ the same for $\mathfrak{K}$. Then the factor-group $\mathfrak{G}/\mathfrak{M}\times\mathfrak{N}$ breaks down into the direct product of two of its subgroups*

$$\mathfrak{G}/(\mathfrak{M}\times\mathfrak{N})\cong\mathfrak{K}/\mathfrak{M}\times\mathfrak{H}/\mathfrak{N}.$$

C o r o l l a r y 2. 1.[5]) *$\mathfrak{G}$ contains, besides $\mathfrak{M}$ and $\mathfrak{N}$, two normal subgroups $\mathfrak{K}'$ and $\mathfrak{H}'$, such that*

$$\mathfrak{H}'/\mathfrak{M}\cong\mathfrak{H},\quad \mathfrak{K}'/\mathfrak{N}\cong\mathfrak{K}.$$

C o r o l l a r y 2. 2.[6]) *We have*

$$\mathfrak{G}/\mathfrak{H}'\cong\mathfrak{K}'/(\mathfrak{M}\times\mathfrak{N}),\quad \mathfrak{G}/\mathfrak{K}'\cong\mathfrak{H}'/(\mathfrak{M}\times\mathfrak{N}).$$

It is easily proved, that if $\mathfrak{G}=\mathfrak{H}\mathfrak{K}$ is a group of semiautomorphic composition, i. e. for example if the permutations for $[\mathfrak{H}]$ are automorphisms of the group $\mathfrak{H}$, then the factor-group $\mathfrak{G}/\mathfrak{M}$ breaks down into the product

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[5]) This follows from theorem 2 when combined with theorem 20 of A. Speiser, *Theorie der Gruppen von endlicher Ordnung*, 3rd edition (Berlin, 1937).

[6]) This follows from corollary 2. 1 when combined with theorem 23 of Speiser, l. c. [5]).

of two of its subgroups

$$\mathfrak{G}/\mathfrak{M} \cong (\mathfrak{K}/\mathfrak{M}) \cdot \mathfrak{H},$$

where $\mathfrak{H}$ is a normal subgroup of the group $(\mathfrak{K}/\mathfrak{M}) \cdot \mathfrak{H}$.

§ 3.

Theorem 3. *If in the finite group $\mathfrak{G} = \mathfrak{H}\mathfrak{K}$ the orders of the groups $\mathfrak{H}$ and $\mathfrak{K}$ are relatively prime, then every normal subgroup $\overline{\mathfrak{G}}$ of $\mathfrak{G}$ is either a normal subgroup of $\mathfrak{H}$ or of $\mathfrak{K}$, or it is of the form $\overline{\mathfrak{G}} = \overline{\mathfrak{H}}\overline{\mathfrak{K}}$ where $\overline{\mathfrak{H}}$ and $\overline{\mathfrak{K}}$ are normal subgroups of $\mathfrak{H}$ and $\mathfrak{K}$, respectively.*

Proof. Let the elements of $\overline{\mathfrak{G}}$ be $H_1K_1, H_2K_2, \ldots$; then

$$H_iK_i \in \overline{\mathfrak{G}} \overset{7)}{\longrightarrow} K_i(H_iK_i)K_i^{-1} = K_iH_i \in \overline{\mathfrak{G}} \longrightarrow H_iK_iK_iH_i = H_iK_i^2H_i \in \overline{\mathfrak{G}} \longrightarrow$$
$$\longrightarrow H_i(H_iK_i^2H_i)H_i^{-1} = H_i^2K_i^2 \in \overline{\mathfrak{G}} \longrightarrow H_i^2K_i^2K_iH_i = H_i^2K_i^3H_i \in \overline{\mathfrak{G}} \longrightarrow$$
$$\longrightarrow H_i(H_i^2K_i^3H_i)H_i^{-1} = H_i^3K_i^3 \in \overline{\mathfrak{G}} \longrightarrow \ldots$$

i. e. $H_iK_i \in \overline{\mathfrak{G}}$ implies $H_i^rK_i^r \in \overline{\mathfrak{G}}$ for $r = 1, 2, \ldots$. If r is the order of the element H_i, then $K_i^r \in \overline{\mathfrak{G}}$, i. e. $K_i \in \overline{\mathfrak{G}}$ (since r and the order of K_i are relatively prime). By a similar reasoning we find $H_i \in \overline{\mathfrak{G}}$. Hence we conclude:

$$\overline{\mathfrak{H}}\overline{\mathfrak{K}} = \overline{\mathfrak{G}} \quad (\overline{\mathfrak{H}} = \{H_1, H_2, \ldots\}, \ \overline{\mathfrak{K}} = \{K_1, K_2, \ldots\}).$$

$H^{-1}\overline{\mathfrak{H}}H \subset \mathfrak{H}$ and $H^{-1}\overline{\mathfrak{H}}H \subset \overline{\mathfrak{G}}$ $(H \in \mathfrak{H})$ imply that $\overline{\mathfrak{H}}$ is a normal subgroup of $\mathfrak{H}$. Similarly, $\overline{\mathfrak{K}}$ is a normal subgroup of $\mathfrak{K}$.

Corollary 3. *If in the finite group $\mathfrak{G} = \mathfrak{H}\mathfrak{K}$, where $((\mathfrak{H}), (\mathfrak{K})) = 1$, the groups $\mathfrak{H}$ and $\mathfrak{K}$ are simple and $[\mathfrak{H}] \cong \mathfrak{K}$, $[\mathfrak{K}] \cong \mathfrak{H}$, then the group $\mathfrak{G}$ is necessarily simple.*

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[7]) The sign $\longrightarrow$ is the sign of implication.