# The meet-decomposition of elements in lattice-ordered semi-groups.

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**1.** In this paper we propose to deal with the structure of elements in a lattice-ordered semi-group, with special emphasis on properties similar to the decomposition theorems of ideal theory. Some concepts of abstract ideal theory are here generalized by using the cardinal notion of *"closure operator"*. The interest of this new method of discussing such problems lies not only in the novelty of the method, but also in the far-reaching generality of the theorems. The two fundamental ideal-theoretic concepts introduced on the basis of closure operator are "*$\Phi$-prime*" and "*$\Phi$-primary*" whose definitions are given in sections 3 and 4, respectively. We shall show how to extend the theory of primary and quasi-primary ideals to cover the new concepts. The theorems remain essentially the same; the proofs are logically complete here, only in some of the applications make we use of well-known results or familiar methods without giving any details.

**2.** Let $G$ be a *commutative l-semi-group,* i. e. a closed[1]) lattice in which a binary commutative and associative multiplication is defined satisfying the distributive law

(2. 1) $$a(b \cup c) = ab \cup ac.$$

Let $G$ have a zero 0 and a unity $e$ such that

(2. 2) $$0 \leq x \leq e \quad \text{for all } x \in G,$$

(2. 3) $$0x = 0, \ ex = x \quad \text{for all } x \in G.$$

It is easy to show that (2. 1) implies the monotony of multiplication:

(2. 4) $$a \leq b \text{ implies } ax \leq bx \text{ for all } x \in G,$$

and this together with (2. 2) and (2. 3) implies

(2. 5) $$a_1 a_2 \ldots a_k \leq a_1 \cap a_2 \cap \ldots \cap a_k \quad \text{for any } k.$$

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[1]) A lattice is called closed if it contains the join and meet of any subset of its elements.

We further assume that $G$ enjoys the *ascending chain condition: no infinite ascending chain $a_1 < a_2 < \dots$ with different terms exists.*

Let $\Phi$ be a *"closure operator"* defined on $G$; that is, $\Phi$ is a mapping $x \to \Phi(x)$ of all elements of $G$ onto a subset of $G$ such that[2])

(i) $\qquad x \leqq \Phi(x) \qquad$ *(extensivity)*;

(ii) $\qquad \Phi(\Phi(x)) = \Phi(x) \qquad$ *(idempotency)*;

(iii) $\qquad x \leqq y$ *implies* $\Phi(x) \leqq \Phi(y) \qquad$ *(monotony).*

$\Phi(x)$ will said to be the *closure of* $x$. If $\Phi(x) = x$, $x$ is called *closed*. For the subset of all closed elements we may write $\Phi(G)$.

It is an elementary fact, used several times, that $x \leqq \Phi(y)$ is equivalent to $\Phi(x) \leqq \Phi(y)$. (Apply (i), (ii) and (iii).)

We shall consider closure operatoıs which are *linear*[3]) in the sense that

(iv) $$\Phi(x \cap y) = \Phi(x) \cap \Phi(y).$$

We observe that (iii) is a simple consequence of (iv), while (iii) implies only $\Phi(x \cap y) \leqq \Phi(x) \cap \Phi(y)$.

*In what follows $\Phi$ will always mean any fixed linear closure operator.*

As is shown in BIRKHOFF [1, p. 201], one may define in $G$ the *residual $a : b$ of $a$ by $b$* as the join of all $x$ with $bx \leqq a$; this residual always exists under our assumptions on $G$. For the residuals one has the following important rules:

(2. 6) $$(\wedge a_n) : b = \wedge (a_n : b),$$

(2. 7) $$a : (\vee b_i) = \wedge (a : b_i),$$

(2. 8) $$a : (bc) = (a : b) : c.$$

After these preliminaries we are now ready to introduce the fundamental concepts mentioned at the beginning.

**3.** An element $p$ of $G$ will be called *$\Phi$-prime,* if $a_1 a_2 \dots a_k \leqq p$ ($k$ arbitrary) implies $a_i \leqq \Phi(p)$ for some subscript $i$. Trivial $\Phi$-primes are $e$ itself and all elements whose $\Phi$-closure is $e$.

Defining the *radical* $r$ of an element $a$ as the join of all $x$ such that $x^n \leqq a$ for some $n = n(x)$, we prove that if $p$ is $\Phi$-prime, then so is its radical $r$. In fact, supposing $a_1 \dots a_k \leqq r$, we have[4]) $a_1^n \dots a_k^n \leqq p$ for some $n$, and hence by the $\Phi$-prime character of $p$, $a_i \leqq \Phi(p)$ for some $i$. Considering that $p \leqq r$, by (iii) we are led to $a_i \leqq \Phi(r)$, as stated.

Examples. $\alpha$) Let identically $\Phi(x) = \iota(x) = x$, that is to say, $\iota$ is the identity operator leaving every element of $G$ unchanged. The $\iota$-prime elements are those which are commonly called primes.

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[2]) *"Closure operator"* is here used in the sense of WARD [1]. — Numbers in brackets refer to the bibliography given at the end of the paper.

[3]) This definition is due to WARD [1]. He has defined four types of linearity, of which our is the third.

[4]) Here we apply the ascending chain condition.

We note the trivial fact that $x$ is necessarily $\Phi$-prime if $\Phi(x)$ is $\iota$-prime. It is further immediate that a $\Phi$-closed element is $\Phi$-prime if and only if it is $\iota$-prime.

$\beta$) Let $\Phi(x)=\varrho(x)$ be the radical of $x$. Then the $\varrho$-primes are the *quasi-primary* elements in the sense that $ab\leq p$ implies that some power of $a$ or of $b$ is $\leq p$ (cf. FUCHS [1]). In this case one can prove that $p$ is $\varrho$-prime if and only if its radical is $\varrho$-prime. For, if $r$ is $\varrho$-prime and $a_1\ldots a_k\leq p\leq r$, then $a_t\leq\varrho(r)=r=\varrho(p)$; (the converse has already been proved in the second paragraph of this section). Since every radical is $\varrho$-closed, it follows the important fact that $p$ is $\varrho$-prime if and only if its radical is $\iota$-prime (cf. FUCHS [1], Definition 2).

$\gamma$) Let $\psi(x)$ be any fixed linear closure operator and define $\mu(x)$ as the meet of all $\psi$-primes which are minimal[5]) in $G$ and contain $x$. It is clear that those elements which are contained in only one minimal $\psi$-prime of $G$ are necessarily $\mu$-primes. The most important and interesting subcase is when $\psi=\iota$; then taking for $G$ the $l$-semi-group of the ideals of a commutative integrity domain with identity, integrally closed in its quotient-field, the $\mu$-primes are those elements which are quasi-equal to some power of an $\iota$-prime (quasi-equality is here to be taken in the sense of VAN DER WAERDEN-ARTIN [6])).

We turn now our attention to the consideration of the meet of $\Phi$-primes.

Theorem 1.[7]) *The meet of a finite number of $\Phi$-primes $p=p_1\cap\ldots\cap p_r$ is $\Phi$-prime again, if and only if one closure $\Phi(p_s)$ is contained in all other $\Phi(p_t)$.*

For, if this condition holds, then by linearity $\Phi(p)=\Phi(p_1\cap\ldots\cap p_r)=$ $=\Phi(p_1)\cap\ldots\cap\Phi(p_r)=\Phi(p_s)$, and $a_1\ldots a_k\leq p$ implies $a_1\ldots a_k\leq p_s$, whence $a_t\leq\Phi(p_s)=\Phi(p)$ for some $i$, proving the assertion.

However, if in the set $\Phi(p_1),\ldots,\Phi(p_r)$ there exist at least two different minimal ones, then $p=p_1\cap\ldots\cap p_r$ is never $\Phi$-prime. To prove this, suppose that $p$ is $\Phi$-prime. By formula (2.5) we get $p_1\ldots p_r\leq p$ and hence conclude that $p_s\leq\Phi(p)$ for at least one subscript $s$, i.e., $\Phi(p_s)\leq\Phi(p)$, and since $\Phi(p)\leq\Phi(p_t)$ for $t=1,2,\ldots,r$, this result proves what is stated in theorem 1.

Consider the set $P$ of the meets of all subsets of $\Phi$-primes in $G$. $P$ is clearly closed under meet and consists, by the ascending chain condition, of all elements in $G$ representable as the finite meet of $\Phi$-primes. For

[5]) Minimal means that it contains no other $\psi$-prime.

[6]) See VAN DER WAERDEN [1], p. 93. — A new closure operation would be $x\to\varkappa(x)$ where $\varkappa(x)$ is the *kernel* of $x$ defined in KRULL [1], or $x\to\varepsilon(x)$, the greatest element quasi-equal to $x$.

[7]) Cf. FUCHS [1], Theorem 1.

example, taking for $G$ the set of all ideals of a commutative ring (with maximal condition), in case $\alpha$) $P$ is equal to the set of half-prime ideals (the set of radicals, cf. KRULL [1]) and in case $\beta$) $P$ exhausts all elements of $G$ (cf. FUCHS [1], Theorem 5).

Theorem 2.[8]) *Assume that* $a \in P$ *and* $a = p_1 \cap \ldots \cap p_r = q_1 \cap \ldots \cap q_s$, *where the components* $p_i$, $q_j$ *are* $\Phi$*-primes. If both decompositions are shortest in the sense that no component may be omitted and no subset of the* $p_i$ *or of the* $q_j$ *has a* $\Phi$*-prime meet, then* $r = s$ *and by proper arrangement we have* $\Phi(p_i) = \Phi(q_i)$

Considering that $p_1 \ldots p_r \leq a \leq q_j$ implies $p_i \leq \Phi(q_j)$ for some $i = i(j)$, we have $\Phi(p_i) \leq \Phi(q_j)$. With this $j$ the same reasoning yields $\Phi(q_j) \leq \Phi(p_k)$ for some $k = k(j)$. Thus $\Phi(p_i) \leq \Phi(q_j) \leq \Phi(p_k)$, implying by theorem 1 that $p_i \cap p_k$ is again $\Phi$-prime. This is contradictory to hypothesis if $i \neq k$; hence $i = k$ and $\Phi(p_i) = \Phi(q_j)$, as stated.

Corollary. *Every element of* $P$ *may be represented as a finite meet of* $\Phi$*-primes, where the closures of the components are uniquely determined.*

**4.** An element $y$ of $G$ will said to be $\Phi$*-primary* if $a_1 \ldots a_k \leq y$ ($k$ arbitrary) implies $a_1 \leq y$ or $a_2 \leq \Phi(y), \ldots,$ or $a_k \leq \Phi(y)$. It is immediate that each $\Phi$-primary element is at the same time $\Phi$-prime and all $y$ satisfying $\Phi(y) = e$ are necessarily $\Phi$-primary.

The reader will readily convince himself that the two concepts: "$\iota$-prime" and "$\iota$-primary" coincide, and that the $\varrho$-primary elements are those which are primary in the ordinary sense.

Theorem 3.[9]) *An irredundant meet of a finite number of* $\Phi$*-primary elements,* $y = y_1 \cap \ldots \cap y_r$ *is again* $\Phi$*-primary, if and only if all* $y_i$ *have the same closure.*

Assume $y = y_1 \cap \ldots \cap y_r$ with $\Phi$-primary components possessing the same $\Phi(y_i)$; $\Phi(y_i) = \Phi(y)$. Further let $a_1 \ldots a_n \leq y$. If none of $a_i$ $(i = 2, \ldots, n)$ is contained in $\Phi(y)$, then $a_1 \ldots a_n \leq y_k$ implies by the $\Phi$-primary character of $y_k$ that $a_1 \leq y_k$. Since this relation holds for all $k$, we have $a_1 \leq y$, indeed.

Let conversely $y = y_1 \cap \ldots \cap y_r$ be an *irredundant* meet of $\Phi$-primary elements. The case where all $\Phi(y_i)$ are different is capable of uniting all $y_i$ with the same $\Phi(y_i)$ into one $\Phi$-primary element, in accordance with what has already been proved. Supposing this case with $r \geq 2$, we may clearly choose a $\Phi(y_i)$, say $\Phi(y_1)$, containing no other $\Phi(y_i)$. If $y$ were $\Phi$-primary, then $y_1 \ldots y_r \leq y$ would imply $y_k \leq \Phi(y)$ for at least one $k \geq 2$, considering that

8) See FUCHS [1], Theorem 6.

9) Cf. e. g. VAN DER WAERDEN [1], pp. 32—24. The term "irredundant" will mean that no component may be omitted. Note that irredundancy is needed only in the "necessary" part of the proof.

by irredundancy $y_1 \leq y$ is impossible. We thus get $\Phi(y_k) \leq \Phi(y) \prec \Phi(y_1)$ for some $k \geq 2$, contrary to the choice of $y_1$. Q. e. d.

Now we define the set $Y$ in the same manner as the set $P$ was defined in the foregoing section, with the sole modification that instead of "$\Phi$-prime" we use the term "$\Phi$-primary". As is readily seen, $Y$ is a subset of $P$ and if $G$ is the set of ideals of a commutative ring, then $Y$ coincides with $G$.

The following theorem will correspond to theorem 2.

Theorem 4.[10]) *Let $a = y_1 \cap \ldots \cap y_n = z_1 \cap \ldots \cap z_m$ be any two shortest decompositions of $a \in Y$ into $\Phi$-primary components. Then $n = m$ and the closures of the components are the same in both decompositions.*

If $\Phi(y_i)$ is maximal among the closures $\Phi(y_1), \ldots, \Phi(y_n)$, $\Phi(z_1), \ldots, \Phi(z_m)$, then one may find a $\Phi(z_j)$ containing $\Phi(y_i)$. For, if this were not so, if e. g. although $\Phi(y_n)$ is maximal none of $\Phi(z_j)$ contained $\Phi(y_n)$, we should have from

$$(4.1) \qquad a : y_n = (y_1 : y_n) \cap \ldots \cap (y_n : y_n) = (z_1 : y_n) \cap \ldots \cap (z_m : y_n)$$

(cf. (2.6)) the relation

$$(4.2) \qquad y_1 \cap \ldots \cap y_{n-1} = z_1 \cap \ldots \cap z_m = a,$$

since $\Phi(y_n)$, and hence $y_n$ is contained in none of $\Phi(y_1), \ldots, \Phi(y_{n-1})$, $\Phi(z_1), \ldots, \Phi(z_m)$, by hypothesis.[11]) Consequently, $y$ would be redundant. This is absurd!

From this fact we conclude at once that the same maximal closures are associated with both representations. When e. g. $\Phi(y_n) = \Phi(z_m)$ is maximal, then with $y = y_n z_m$ by (2.6) and (2.8) we get

$$(4.3) \qquad a_1 = a : y = (y_1 : y) \cap \ldots \cap (y_n : y) = y_1 \cap \ldots \cap y_{n-1}$$

and on the other hand

$$(4.4) \qquad a_1 = (z_1 : y) \cap \ldots \cap (z_m : y) = z_1 \cap \ldots \cap z_{m-1}.$$

Our theorem is now by induction completely proved.

In addition, whenever $\Phi(y_1) = \Phi(z_1)$ is a minimal one in the set of the $\Phi(y_i)$, then $y_1 = z_1$. In other words, this means that the isolated $\Phi$ primary components of $a$, i. e., those associated with a minimal $\Phi(y_i)$, are unique.

For the proof we first observe that if $y$ is $\Phi$-primary and none of the closures associated with the $\Phi$-primary representations of $b \in Y$ is $\leq \Phi(y)$, then $y : b = y$. In fact, if $b$ is $\Phi$-primary, the statement is evident; if not, suppose $b = x_1 \cap \ldots \cap x_k$ is a $\Phi$-primary decomposition of $b$. Then using (2.8) repeatedly we have by hypothesis

[10]) See Noether [1], p. 44, or van der Waerden [1], pp. 35—36.

[11]) We have made use of the simple fact that if $y$ is $\Phi$-primary and $a$ is not $\leq \Phi(y)$, then $y : a = y$, this being an immediate consequence of the definition of $\Phi$-primary elements.

$$(4.5)\quad y \leqq y : b = y : (x_1 \cap \ldots \cap x_k) \leqq y : (x_1 \ldots x_k) = \\ = (y : x_1) : (x_2 \ldots x_k) = y : (x_2 \ldots x_k) = \ldots = y$$

proving the assertion.

Now the minimality of $\Phi(y_1) = \Phi(z_1)$ implies that none of the closures associated with the $\Phi$-primary representations of $c = y_2 \cap \ldots \cap y_n \cap z_2 \cap \ldots \cap z_m$ is $\leqq \Phi(y_1)$, consequently,

$$(4.6)\qquad a : c = (y_1 : c) \cap (y_2 : c) \cap \ldots \cap (y_n : c) = y_1 \cap e \cap \ldots \cap e = y_1$$

and similarly $a : c = z_1$, whence $y_1 = z_1$ as we wished to prove.

The following corollary is immediate.

Corollary. *Any element of $Y$ has a decomposition into the meet of $\Phi$-primary elements: in two such decompositions the closures of the components as well as the isolated components themselves are necessarily the same.*

**5.** It is a matter of some interest to have information about an interesting connection between $\Phi$-prime and $\Phi$-primary elements.

Define $q$ *$\Phi$-maximal*, if while $\Phi(q) \neq e$, $\Phi(q) < \Phi(x)$ implies $\Phi(x) = e$. Then we have

Theorem 5. *If the only element $x$ with $\Phi(x) = e$ is $x = e$, then every $\Phi$-maximal $\Phi$-prime element is $\Phi$-primary.*

Assume $p$ $\Phi$-maximal and $\Phi$-prime, and $a_1 \ldots a_n \leqq p$, where $a_i$ is not $\leqq \Phi(p)$ for $i \geqq 2$. We have to show that $a_1 \leqq p$. Now $\Phi(p \cup a_i) \geqq \Phi(p) \cup \Phi(a_i) > \Phi(p)$, (because $\Phi(a_i)$ is not $\leqq \Phi(p)$), and hence $\Phi(p \cup a_i) = e$ for $i \geqq 2$, by the $\Phi$-maxi mality of $p$. By hypothesis, $p \cup a_i = e$ for $i \geqq 2$. From $a_1 \ldots a_n \leqq p$ now it follows by (2. 1) that

$$(5.1)\qquad a_1(p \cup a_2) \ldots (p \cup a_n) \leqq p$$

and as $p \cup a_i = e$ for $i \geqq 2$, we are directly led to $a_1 \leqq p$, q. e d.

In particular, when $G$ is the set of ideals of an integrity domain, then theorem 5 expresses the fact that if $e$ is the sole element with radical $e$ (this means that the ring has a unit element), then the quasi-primary ideals with minimal prime radicals are simply primary (Fuchs [1], Theorem 8).

**6.** We conclude by giving the following notion[12]).

We shall call $a$ *$\Phi$-primary to $b$* if $b : a \leqq \Phi(b)$. In case $b$ is $\Phi$-primary, just the elements $x$ satisfying $x \leqq b$ are $\Phi$-primary to $b$. Although it is quite clear it seems to be worth while noticing that "*$a$ is $\iota$-primary to $b$*" means nothing else, than $a$ is prime to $b$ in the common sense defined by Noether [1], p. 45; cf. also van der Waerden [1], p. 25.

Assume $b = y_1 \cap \ldots \cap y_n$ is a shortest $\Phi$-primary decomposition of $b \in Y$, where $y_1, \ldots, y_k$ $(k \leqq n)$ are isolated, $y_{k+1}, \ldots, y_n$, if any, are not isolated

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[12]) The concept "*$a$ is primary to $b$*" is defined in Fuchs [2], cf. also Fuchs [4]. A generalization of it, essentially equivalent to $\Phi$-primarity, may be found in Fuchs [3].

$\Phi$-primary components of $b$. If $a$ is not $\leqq y_j$ for $j=1, \ldots, k$, or, what is the same, if $y_j : a \leqq \Phi(y_j)$ for $j \leqq k$, then $a$ is $\Phi$-primary to $b$. In fact, hypotheses imply

(6. 1) $\quad b : a = (y_1 : a) \cap \ldots \cap (y_n : a) \leqq \Phi(y_1) \cap \ldots \cap \Phi(y_k) \cap e \cap \ldots \cap e = \Phi(b).$

If we impose a further restriction on the $\Phi$-operation, namely,

(*) *if $y, y_1, y_2$ are $\Phi$-primary elements, then $\Phi(y_1) \cap \Phi(y_2) \leqq \Phi(y)$ implies $\Phi(y_1) \leqq \Phi(y)$ or $\Phi(y_2) \leqq \Phi(y)$,*[13])

we can prove even the converse of our last statement: *For $b : a \leqq \Phi(b)$ it is necessary that $a$ be not $\leqq y_j$ for $j \leqq k$.* To verify this, suppose $b : a \leqq \Phi(b)$ and, say, $a \leqq y_1$. Then

$$a(y_2 \cap \ldots \cap y_n) \leqq a \cap y_2 \cap \ldots \cap y_n \leqq b$$

and hence by hypothesis we get

$$y_2 \cap \ldots \cap y_n \leqq \Phi(b) \leqq \Phi(y_1).$$

In view of our new restriction (*) on $\Phi$, we conclude, using a simple induction, that $\Phi(y_i) \leqq \Phi(y_1)$ for some $i \geqq 2$, contrary to the isolated character of $y_1$. To sum up, we have proved

Theorem 6. *Supposing (*), $a$ is $\Phi$-primary to $b \in Y$, if and only if $a$ is $\Phi$-primary to all isolated $\Phi$-primary components of $b$.*

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[13]) This restriction is satisfied in all important special cases.