On the imbedding of n-dimensional sets in $2n$-dimensional absolute retracts.

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1. Under "imbedding theorems"[1]) one understands theorems dealing with the possibility of homeomorphic mapping of spaces of some kind onto subsets of some other spaces of more regular properties.

From the homological point of view, a space has a regular structure if it is compact and if its groups of homology have a simple algebraic structure. From the homotopic point of view, the simple structure of the fundamental group is also an important requirement, and so is also the regularity of the local structure, guaranteed, for instance, by local contractibility[2]).

Thus we may regard as the simplest point sets those local contractible compacta for which the groups of homology and the fundamental group consist only of the identity. For spaces of finite dimension these conditions characterise the absolute retracts[3]). Hence the problem of imbedding a space in a space of possible simplest homological and homotopic structure and smallest dimension can be formulated as follows:

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[1]) Such are for instance: 1. the classic theorem of Urysohn that every metric separable space is topologically contained in the Hilbert cube, 2. the Menger—Nöbeling theorem that every n-dimensional metric separable space is homeomorphic to a subset of the cartesian $(2n+1)$-dimensional space C_{2n+1}, 3. the theorem that every metric separable space is topologically contained in a compact space of the same dimension, 4. the theorem that every metric separable space of positive dimension is homeomorphic to a subset of a Peano space of the same dimension.

[2]) A space M is said to be *locally contractible* in the point $a \in M$ if every neighborhood U of a contains another neighborhood V which is contractible in U, i. e. such that there exists a continuous mapping $f(x, t)$ defined for $x \in V$ and $0 \leqq t \leqq 1$ and satisfying to the conditions $f(x, 0) = x$, $f(x, t) \in U$, $f(x, 1) = a$ for every $x \in V$ and $0 \leqq t \leqq 1$. Every polytope is locally contractible. Every locally contractible space is locally connected in all dimensions.

[3]) A continuous mapping f of a space M onto its subset E is said to be a *retraction*, if $f\ x) = x$ for every $x \in E$. Then the set E is called the *retract* of M. The compactum A is said to be an *absolute retract* if it is a topological image of a retract of the Hilbert cube. A necessary and sufficient condition for a compactum A of finite dimension to be an absolute retract is that A be locally contractible, acyclic in all dimensions and that the fundamental group of A consists only of the identity. See W. HUREWICZ, Beiträge zur Topologie der Deformationen, I. Höherdimensionale Homotopiegruppen, *Proceedings Academy Amsterdam*, **38** (1938), p. 113.

What is the smallest number m such that every metric separable space of dimension n is topologically contained in an absolute retract of dimension m?

By the imbedding theorem of MENGER and NÖBELING[4]), every metric separable n-dimensional space is homeomorphic to a subset of the cartesian $(2n+1)$-dimensional simplex. Hence $m \leqq 2n+1$.

The purpose of this paper is to show that $m \leqq 2n$ provided that n is positive[5]).

2. Lemma 1. *Let Δ_0 be an $(n-1)$-dimensional simplex lying in the cartesian n-dimensional space C_n and L a polygonal simple arc having the barycentric center a_0 of Δ_0 as one of its ends. If $\Delta_0 \cdot L = a_0$, then there exists a simplicial homeomorphism h of C_n into itself such that $h(x) = x$ for every $x \in \Delta_0$ and that $h(L)$ is a segment perpendicular to Δ_0.*

Proof. Let k denote the number of segments constituting L. If $k=1$, then L is a segment and there exists an affine transformation mapping the $(n-1)$-dimensional hyperplane containing Δ_0 by identity and the segment L into a segment perpendicular to Δ_0. Let us assume that $k>1$ and that the lemma is valid for polygonal arcs constituted by $<k$ segments. Let a_1 denote the end of L different from a_0 and let $L_1, L_2, \ldots, L_k$ be all segments of L in the order in which they occur in L from a_1 to a_0. We can assume that, for $i=1, 2, \ldots, (k-1)$, the segment L_{i+1} is not a straight line prolongation of the segment L_i. Let a_2 be the common end of the segments L_1 and L and let a_3 denote a point lying on the straight line prolongation of the segment L_2 beyond the end a_2 so near to a_2, that the common part of L and the triangle $\Delta(a_1, a_2, a_3)$ is L_1.

Let H denote an $(n-1)$-dimensional hyperplane passing through the segment $\overline{a_1 a_3}$ but not containing the point a_2. It is clear that there exists in H an $(n-1)$-dimensional simplex $\Delta_1 = \Delta(b_1', b_2', \ldots, b_n')$ containing $\overline{a_1 a_3}$ in its interior and such that the intersection of the n-dimensional simplex $\Delta_2' = \Delta(a_2, b_1', b_2', \ldots, b_n')$ with the set $L + \Delta_0$ is the segment L_1. Let us choose, for every $i=1, 2, \ldots, n$, a point b_i lying on the straight line prolongation of the segment $\overline{a_2 b_i'}$ beyond the end b_i'. It is easy to see that, provided that the distances $\varrho(b_i, b_i')$ are sufficiently small, the n-dimensional simplex $\Delta_2 = \Delta(a_2, b_1, b_2, \ldots, b_n)$ satisfies the following conditions: 1) $\Delta_2 \cdot (L + \Delta_0) = L_1$, 2) $L_1 - (a_2)$ lies in the interior of Δ_2.

Let us decompose Δ_2 into $n+1$ n-dimensional simplexes having a_1 as their common vertex and the $(n-1)$-faces of Δ_2 as their bases. Putting

$$\varphi(a_1) = a_3, \ \varphi(a_2) = a_2 \text{ and } \varphi(b_i) = b_i, \text{ for } i = 1, 2, \ldots, n,$$

4) See, for instance, W. HUREWICZ and H. WALLMAN, *Dimension Theory* (Princeton, 1941), p. 60.

5) Cf. my paper: Sur le plongement des espaces dans les rétractes absolus, *Fundamenta Math.*, **27** (1936), p. 242, where I expressed the conjecture that $m = n+1$.

we define a simplicial transformation φ of Δ_2 in itself such that it is the identity on the boundary of Δ_2 and maps the segment L_1 onto the segment $\overline{a_2a_3}$. If we put

$$\varphi(x)=x, \text{ for every } x\in C_n-\Delta_2,$$

we obtain a simplicial homeomorphism φ which is the identity on Δ_0 and maps the polygonal arc L onto the polygonal arc

$$L'=(\overline{a_2a_3}+L_2)+L_3+\ldots+L_k.$$

But $\overline{a_2a_3}+L_2$ is a segment, hence the polygonal arc L' consists of $k-1$ segments. By the hypothesis of induction applied to Δ_0 and L' there exists a simplicial homeomorphism ψ being the identity on Δ_0 and mapping L onto a segment perpendicular to Δ_0. If we put

$$h(x)=\psi\varphi(x) \qquad \text{for every } x\in C_n,$$

we obtain the desired simplicial homeomorphism h of C_n into itself mapping L onto a segment perpendicular to Δ_0 and satisfying the condition $h(x)=x$ for every $x\in\Delta_0$.

3. Lemma 2. *Let P be an m-dimensional strongly connected polytope lying in the m-dimensional $(m>1)$ cartesian space C_m and let T be a triangulation of P. If Δ_0 is an $(m-1)$-dimensional simplex of T lying on the boundary B of P and E a compact subset of $P-\Delta_0$ such that $\dim E<m-1$, then there exists a simplicial retraction $r(x)$ of P satisfying the following conditions:* 1) $E+B-\Delta_0\subset r(P)$, 2) *for every m-dimensional simplex Δ of the triangulation T: $\Delta\cdot(P-r(P))\neq 0$.*

Proof. In every m-dimensional simplex of the triangulation T let us choose an interior point belonging to $P-E$. Thus we obtain a finite system of points $a_1, a_2, \ldots, a_k$. Since P is strongly connected, there exists a polygonal simple arc L such that

1. L has as one of its ends the barycentric center a_0 of the simplex Δ_0,
2. $L-(a_0)\subset P-B-E$,
3. $a_i\in L$ for every $i=1,2,\ldots,k$.

By lemma 1, there exists a simplicial homeomorphism h mapping C_m on itself in such a manner that $h(x)=x$ for every $x\in\Delta_0$ and that $h(L)$ is a segment perpendicular to Δ_0. The point a_0 is one of the ends of the segment $h(L)$. Let b_0 be the other end of $h(L)$. The set $P_1=h(P)$ is a strongly connected polytope containing $h(L)-(a_0)$ in its interior and Δ_0 on its boundary. Let Δ_0' denote an $(n-1)$-dimensional simplex contained in Δ_0 and being a neighborhood of a_0 in Δ_0 so small that for every $x\in\Delta_0'$ the segment $\overline{xb_0}$ lies in the interior of P_1. The sum of all segments $\overline{xb_0}$ with $x\in\Delta_0'$ is an m-dimensional simplex Δ_1 having Δ_0' as one of its faces. Let us denote by Λ the sum of all faces of Δ_1 different from Δ_0'. Consider, for every $y\in\Delta_1$, the straight line Γ_y passing through y and perpendicular to Δ_0. Clearly Γ_y cuts

$\varDelta$ in one point; let us denote this point by $r_1(y)$. Moreover, if we put $r_1(y)=y$ for every $y \in P_1 - \varDelta_1$, we obtain a simplicial retraction r_1 of P_1 to the closure of the set $P_1 - \varDelta_1$. Putting

$$r(x) = h^{-1} r_1 h(x) \qquad \text{for every } x \in P,$$

we obtain the required retraction.

4. Theorem. *Every metric separable n-dimensional space E is homeomorphic to a subset of some absolute retract of dimension* $\leqq 2n$ *lying in the cartesian* $(2n+1)$*-dimensional space* C_{2n+1}.

Proof. By the Menger-Nöbeling imbedding theorem we can assume that E is a subset of a $(2n+1)$-dimensional simplex $\varDelta$ having the diameter $\leqq 1$. Let us denote by T_k the result of the process of barycentric subdivision when iterated k times. The diameters of all simplexes of the triangulation T_k are $\leqq \left(\frac{2n+1}{2n+2}\right)^k$ [6]).

Now let us define the sequence $\{r_k\}$ of retractions as follows:

Putting $r_0(x)=x$ for every $x \in \varDelta$, let us assume that the retraction $r_k(x)$ mapping $\varDelta$ into a polytope $r_k(\varDelta) \supset E$ has been already defined for some k in such a manner that no one of the $(2n+1)$-dimensional simplexes belonging to T_k is contained in $r_k(\varDelta)$.

If $\varDelta_1^{(k)}, \varDelta_2^{(k)}, \ldots, \varDelta_{m_k}^{(k)}$ are all $(2n+1)$-dimensional simplexes of the triangulation T_k, then

$$r_k(\varDelta) = \sum_{\nu=1}^{m_k} \varDelta_\nu^{(k)} \cdot r_k(\varDelta).$$

Every of the sets $\varDelta_\nu^{(k)} \cdot r_k(\varDelta)$, $\nu = 1, 2, \ldots, m_k$ is a polytope. Hence there exists a polytope $P_0(\nu, k)$ of dimension $\leqq 2n$ and a system of strongly connected $(2n+1)$-dimensional polytopes $P_1(\nu, k), P_2(\nu, k), \ldots, P_{\alpha_{\nu,k}}(\nu, k)$ such that

$$\varDelta_\nu^{(k)} \cdot r_k(\varDelta) = P_0(\nu, k) + P_1(\nu, k) + \ldots + P_{\alpha_{\nu,k}}(\nu, k)$$

and that, for $i \neq j$

$$\dim P_i(\nu, k) \cdot P_j(\nu, k) \leqq 2n - 1.$$

Consider a simplicial decomposition $T_i(\nu, k)$ of $P_i(\nu, k)$, which is a subdivision of the triangulation T_{k+1}. Since $P_i(\nu, k) \subsetneqq \varDelta_\nu^{(k)}$, there exists in the triangulation $T_i(\nu, k)$ a $2n$-dimensional simplex $\varDelta_i(\nu, k)$ lying on the boundary $B_i(\nu, k)$ of $P_i(\nu, k)$, but not contained in the boundary of $\varDelta_\nu^{(k)}$. Applying the lemma 2 we infer that there exists a simplicial retraction $\varphi_i^{(\nu, k)}$ of the polytope $P_i(\nu, k)$ satisfying to following conditions:

1. $E \cdot P_i(\nu, k) + B_i(\nu, k) - \varDelta_i \ \nu, k) \subset \varphi_i^{(\nu, k)}(P_i(\nu, k))$.
2. No one of the $(2n+1)$-dimensional simplexes of the triangulation $T_i(\nu, k)$ is contained in $\varphi_i^{(\nu, k)}(P_i(\nu, k))$.

[6]) See, for instance, P. Alexandroff and H. Hopf, *Topologie*. I (Berlin, 1935), p. 136.

Hence for different systems of indices i, ν and j, μ we have

$$P_i(\nu, k)\cdot(P_j(\mu, k) \subset B_i(\nu, k) - \mathrm{Int}[\Delta_i(\nu, k)],$$

where $\mathrm{Int}[\Delta_i(\nu, k)]$ denotes the interior of the simplex $\Delta_i(\nu, k)$. Hence $\varphi_i^{(\nu,k)}(x) = x$ for every $x \in P_i(\nu, k)\cdot P_j(\mu, k)$. It follows that if we put $\varphi(x) = \varphi_i^{(\nu,k)}(x)$ for $x \in P_i(\nu, k)$, we obtain a simplicial retraction φ of the polytope $r_k(\Delta)$. Putting $r_{k+1}(x) = \varphi r_k(x)$ for every $x \in \Delta$, we obtain a simplicial retraction of Δ into the polytope $\varphi r_k(\Delta)$ such that no one of the $(2n+1)$-dimensional simplexes belonging to the triangulation T_{k+1} is contained in $\varphi r_k(\Delta)$. Moreover, by our construction the points $r_k(x)$ and $r_{k+1}(x) = \varphi r_k(x)$ are points of one of the simplexes of the triangulation T_k. Hence

$$\varrho(r_k(x), r_{k+1}(x)) \leqq \left(\frac{2n+1}{2n+2}\right)^k$$

for every $x \in \Delta$. It follows that the sequence $\{r_k\}$ is uniformly convergent to a continuous transformation r of Δ. Evidently,

$$(1) \qquad r(\Delta) = \prod_{k=1}^{\infty} r_k(\Delta) \text{ and } r(x) = x \text{ for every } x \in r(\Delta).$$

Hence r is a retraction of Δ to an absolute retract $r(\Delta) \supset E$. Moreover, by (1) and the construction of $r_k(\Delta)$, we infer that no one of the $(2n+1)$-dimensional simplexes belonging to T_k $(k = 1, 2, \ldots)$ is contained in $r_k(\Delta)$. Since the diameters of the simplexes belonging to T_k are $\leqq \left(\frac{2n+1}{2n+2}\right)^k$ it follows that the dimension of $r(\Delta)$ is $< 2n+1$. Thus the proof of the theorem is achieved.

4. Corollary. *There exists a $2n$-dimensional absolute retract such that every metric separable space of dimension $\leqq n$ is topologically contained in it.*

Applying the last theorem to the universal n-dimensional compact space M_n of Menger[7]) which contains topologically every metric separable space of dimension $\leqq n$, we obtain a $2n$-dimensional absolute retract with the required property.

Let us remark that M_1 cannot be imbedded in a 2-dimensional polytope. Indeed, M_1 is not contained obviously in no 1-dimensional polytope. Hence if M_1 is homeomorphic to a subset of some triangulated 2-dimensional polytope, there exists an open subset G of M_1 homeomorphic to a subset of the cartesian plane C_2. But this is impossible, because G contains topologically every curve.

(Received November 6, 1949.)

[7]) K. Menger, Über umfassendste n-dimensionale Mengen, *Proceedings Academy Amsterdam,* **29** (1926), p. 1125–1128.