On a geometrical extremum problem.

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1. In what follows we shall consider an extremum problem concerning polygons.[1]) Throughout this paper U_n will mean a convex polygon with n sides each of which has the length 1. The diameter of the polygon, i. e. the longest diagonal, will be discussed.

Trying to find a U_n with the smallest possible diameter, P. ERDŐS found in the cases $n=4$ and 5 that the extemal figures are only the corresponding regular ones. It was expected that this is true generally; but ERDŐS surprisingly found this not be true for $n=6$. The diameter of the regular hexagon is 2, while the hexagon, the angles of which are alternately $\pi/2$ and $3\pi/5$, has a diameter $\sqrt{2+\sqrt{3}}<2$.

It would be interesting to find for every n the polygons U_n with the *minimal diameter* Δ_n. The results of this paper show that the answer to this question depends upon the numbertheoretical structure of n. Our answers are not complete.

As to the part of the question regarding the value of Δ_n, we obtain that if n has at least one odd prime factor, then Δ_n equals to the radius of the circumscribed circle of a regular U_{2n}, i. e. we have

Theorem 1. *If* $n=(2k+1)2^s$, *where* $k\geqq 1$, $s\geqq 0$, *then*

$$\Delta_n=\left(2\sin\frac{\pi}{2n}\right)^{-1}. \tag{1}$$

Thus the problem of the minimum remains open only if $n=2^s>4$. We have for all n the

Theorem 2. *If* $n\geqq 3$, *then*

$$\Delta_n\geqq\left(2\sin\frac{\pi}{2n}\right)^{-1}. \tag{2}$$

As an upper estimation of the value of Δ_n $(n\geqq 3)$ we have

$$\Delta_n\leqq\left(\sin\frac{\pi}{n}\right)^{-1}, \tag{3}$$

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[1]) *Added in proof:* After my paper was finished, I have read the paper of K. REINHARDT, Extremale Polygone gegebenen Durchmessers, *Jahresbericht der Deutschen Math.-Vereinigung*, 31 (1922), pp. 251—270, which deals with a nearly related subject and contains many of my results. Nevertheless I think my paper has some proper interest because of its different point of view and treatment.

Δ_n being evidently at most as large as the diameter of the regular n-gon, which again is not larger than the diameter of its circumscribed circle. This remark as well as our theorem 2 is of significance only if $n = 2^s$. In case of $s \geqq 3$ I did not succeed in deciding whether or not the sign of equality can be reached in estimation (2) or (3). If $s = 3$ something more can be said, namely

Theorem 3. $\Delta_8 < \left(\sin\frac{\pi}{8}\right)^{-1}$, *i. e. the diameter of the regular octagon does not give the minimum belonging to* $n = 8$.

It seems likely that this holds also for $n = 2^s > 8$.

As to the question of unicity of the extremal figure the answer is generally negative. In this respect the dependence upon the numbertheoretical structure of n is more conspicuous. This is clearly shown by the following

Theorem 4. *If the decomposition of n into prime-factors contains at least two odd prime-factors (equal or not), then there are at least two essentially different extremal polygons* U_n. *If* $n = 2k+1$, *then the regular n-gon is among the extremal polygons; if* $n = (2k+1)\cdot 2^s$, $k \geqq 1$, $s \geqq 1$, *it is not.*

In the first mentioned case, when $n = p\cdot q\cdot n'$ (p, q being primes, n' an integer), I shall show that forming the so-called Reuleaux-polygons[2]) with p resp. pq vertices, they can be completed by new points on the periphery into n-gons in such a way that they form extremal U_n's. I could not find *all* extremal polygons so far, if $n > 6$.

2. If the general question is raised, which convex closed curves with a given length l of periphery have the minimal diameter, the well known answer is given by the following formula

$$l = \frac{1}{2}\int_0^{2\pi} B(\varphi)\, d\varphi, \tag{4}$$

where $B(\varphi)$ means the distance between two parallel lines of support, both belonging to the φ-direction of the convex curve. As

$$D = \max_{(\varphi)} B(\varphi) \geqq \frac{1}{2\pi}\int_0^{2\pi} B(\varphi)\, d\varphi = \frac{l}{\pi}$$

holds for the diameter D of the curve, D takes its minimal value for, and only for a curve of constant width.

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[2]) See for ex. T. Bonnesen—W. Fenchel, *Theorie der konvexen Körper* (Berlin, 1934), p. 130.

3. The inequalities (2) and (3) lead to the inequality

$$\frac{\frac{\pi}{n}}{\sin\frac{\pi}{n}} \leq \pi\frac{\Delta_n}{n} \leq \frac{\frac{\pi}{2n}}{\sin\frac{\pi}{2n}},$$

from which it follows for the asymptotical value of the minimal diameter Δ_n that

$$\lim_{n\to\infty}\frac{\Delta_n}{n} = \frac{1}{\pi}.$$

4. Now we turn to the proof of our theorem 2, since theorems 1 and 4 can be deduced without difficulty from it. That our extremal problem has at least one solution, it follows easily by using a classical argument.

For the proof of our theorem 2 we shall need the following:

Theorem 5. *The necessary condition for a polygon U_n being a minimal figure is that each vertex should have an opposite vertex, i. e. a vertex in the distance equal to the diameter.*

We shall prove this theorem in the next paragraph, for the moment let us assume that it has been proved.

Remark: It follows from the example constructed by ERDŐS in the case $n=6$ that our condition is not sufficient. The regular hexagon possesses the above mentioned property, but it is no minimal figure.

We shall use also the following theorem[3]):

Any set with the diameter Δ may be completed to form a domain, the boundary of which is a curve of constant width, with the same diameter.

Finally we shall use the following theorem[4]):

Any closed domain the boundary of which is a curve of constant width Δ, contains together with two of its points P_1 and P_2 all circular arcs passing across P_1 and P_2, which are smaller than a half circle and the radius of which is $\geq \Delta$.

To prove our theorem 2, let us now consider a polygon which is a minimal figure with the diameter Δ_n. Let us complete it in some way to form a domain with the boundary G a curve of constant width. G has the diameter resp. width Δ_n and periphery $\pi\Delta_n$. We prove that every vertex of the polygon is a point of curve G. Assuming that vertex A does not lie on curve G, let us consider the opposite vertex A' of A and continue $\overrightarrow{A'A}$ in this direction. This line would intersect G at the point A'' for which $\overline{A'A''} > \Delta_n$ would hold.

We denote by J_i the part of G between the ith and $(i+1)$th vertex A_i resp. A_{i+1} i. e. the vertices of the minimal figure. The length of J_i shall

[3]) See loc. cit. [2]), p. 130.
[4]) See loc. cit. [2]), p. 129.

be s_i. But the closed domain determined by G contains with its points A_i and A_{i+1} the circular arc C_i the radius of which is $\varDelta_n$ (according to theorem[3])). We observe that in the circle of radius $\varDelta_n$ the opening of the central angle belonging to the arc $\widehat{A_iA_{i+1}}$ is evidently independent of i; it may be denoted by ω. Now the length of the circular arc is $\omega\varDelta_n$. Let us now consider the convex, closed curves L_1 and L_2. L_1 consisting of the arc J_i and of A_iA_{i+1}, while L_2 of C_i and of $\overline{A_iA_{i+1}}$. Evidently L_1 contains L_2; hence according to a well-known theorem the periphery of L_1 is at least as large as the periphery of L_2; or in other words the length of the circular arc is not longer than that of the arc J_i: $\omega\varDelta_n \leqq s_i$. If these inequalities are summed with regard to the index i, we obtain $n\omega\varDelta_n \leqq \sum_1^n{}' s_i = \pi\varDelta_n$, thus

$$\omega \leqq \frac{\pi}{n}. \tag{6}$$

But in consequence of the definition of ω, we have

$$\sin\frac{\omega}{2} = \frac{1}{2\varDelta_n}, \quad \varDelta_n = \left(2\sin\frac{\omega}{2}\right)^{-1},$$

thus, by (5),

$$\varDelta_n \geqq \left(2\sin\frac{\pi}{2n}\right)^{-1},$$

which proves our theorem 2.

5. To prove our theorems 1 and 4 we proceed as follows. From theorem 2 it obviously follows that the value of $\varDelta_n$ cannot be smaller than $\left(2\sin\frac{\pi}{2n}\right)^{-1}$. Hence theorem 1 will be proved if we can show that this lower bound is actually attained if $n=(2k+1)2^s$. In the case $n=2k+1$ this follows simply taking a regular n-gon; this shows at the same time that in this case the regular n-gon is one of the the extremal n-gons. If n is even, the diameter d_n of the regular n-gon is

$$d_n = \left(\sin\frac{\pi}{n}\right)^{-1} > \left(2\sin\frac{\pi}{2n}\right)^{-1};$$

hence if we show, in the case $n=(2k+1)2^s$. $k \geqq 1$, $s \geqq 1$, that for an other polygon U_n the lower bound $\left(2\sin\frac{\pi}{2n}\right)^{-1}$ can be attained, this will show that $\left(2\sin\frac{\pi}{2n}\right)^{-1}$ is the minimal value also in that case and that the corresponding regular n-gon is *not* among the extremal polygons.

Let p be an odd factor of n. Let us consider the p-sided Reuleaux-polygon with the constant width, i. e. with the diameter $\varDelta_n = \left(2\sin\frac{\pi}{2n}\right)^{-1}$.

From any of its vertices the opposite circular arc can be seen at an angle $\frac{\pi}{p}$. We can inscribe in this arc a broken line consisting of $\frac{\pi}{p}:\frac{\pi}{n}=\frac{n}{p}$ sides of length 1. The union of these broken lines forms a polygon U_n with diameter Δ_n. Taking different odd factors of n and applying the same procedure we obtain different Reuleaux-polygons, i. e. different minimal U_n polygons. This completes the proof of our theorem 1.

6. Now we turn to the proof of our theorem 5. For this purpose we need the following four lemmas:

Lemma 1. *If parallel lines of support can be drawn through the end points of a chord of a convex plane curve, then this chord intersects all diameters.*

Let the chord C have the above mentioned property and let the parallel lines of support passing through its endpoints be s_1 and s_2. If a diameter D would exist which did not intersect C, then D would lie in one of the areas determined by s_1, s_2 and C. D being a diameter the normals passing through its endpoints are lines of support, i. e. they contain the curve and consequently C. It is easy to see that this is only possible if C and D have at least one common endpoint.

Lemma 2. *Consider two angles α and β issued from a point E of a straight line e and lying in the same halfplane determined by e, and having no common points except E. If we turn α round E towards β (resp. in the opposite direction, but in the same halfplane) so that they should have no common point except E, then each point of α will get nearer (resp. farther) to each point of β except E itself.*

Let the point A of the angle α be denoted in its new position by A' and let B be a point of β. Let a be the normal of the distance $\overline{AA'}$ in its centre. Then all points which are nearer to A' than to A form the halfplane determined by a and containing A'. But at the same time all points of β are contained there too, q. e. d.

Lemma 3. *If V_n is a minimal figure having an angle π, then we can construct an other minimal figure V_n', all angles of which are $<\pi$.*

If the angle of V_n in the vertex A is π, then A can not have an opposite vertex, for the opposite vertex of A would be at a greater distance from one of the two neighbouring vertices of A. Let A_1A and AA_2 be the two sides of V_n, which form an angle π in the point A, and let A' be the vertex through which a line of support parallel to $\overline{A_1A_2}$ passes. According to lemma 1 all diameters of the polygon intersect the chord AA', therefore the endpoint of the diameters lie on both sides of this chord. It must be remarked, that the part of V_n between A_1 and A', which does not contain A, as well as its part between A' and A_2, which does not contain A, lie in such angles, which are on one side of the supporting line through A'. One of these angles is determined for example by the line $A'A_1$ and by that side of V_n starting from A' which lies on the same side of $A'A$ as A_1. Let us now insert hinges at the vertices A_1, A, A_2, A'. Let us furthermore fix the vertex A' and the line of support going through it, and let us shift the vertex A in the direction of the normal of A_1A_2 and away from A'. According to our lemma 2, all diameters except those starting from vertex A' will decrease.

In the following we may assume that two sides of a minimal figure do not lie on one line, i. e. by applying our hinge method the convexity will not be violated.

Lemma 4. *In the case of minimal figures not all diameters start from the same vertex.*

Let us suppose that all diameters would start from the vertex A. Let the order (in a certain direction) of the opposite vertices be $A_1, A_2, \ldots, A_r$. Let the vertex preceeding A_1 in the mentioned sense be A', and the vertex succeeding A_r be A''. (In consequence of the trivial fact that in the case of $n > 3$, $\Delta_n > 1$, A' and A'' cannot coincide with A.) Let us insert hinges at the vertices A, A', A_1, A_r, A''. Let the maximum of all diagonals, the length of which differs from Δ_n, be $\delta_n < \Delta_n$. Let us now fix vertex A and shift the distance A_1A_r in the direction of its own normal towards A. Then each point A_i gets into such a new position A'_i for which $\overline{A'_iA} \leq \Delta'_n < \Delta_n$. Although there will be such diagonals which still increase, if the change of position is small enough, we see immediately that the maximum of the diagonals not starting from A will not surpass a value $\delta'_n < \Delta'_n$. This would contradict to our assumption that the original polygon is a minimal figure.

To prove our theorem 5 let us now assume that the minimal figure has a vertex A which has no opposite vertex. Let us consider a line of support in A and a line of support through A' parallel to the former one. Let us further consider the vertices A_1 and A_2 neighbouring A. According to our lemma 1, AA' intersects all diameters. Consequently their endpoints lie on opposite sides of AA' and at least one of them may be in A'. We remark further that the parts of the polygon lying between the vertices A' and A_1, resp. A' and A_2, neither of which contains A, are contained in two angels, both in conformity to the assumption of our lemma 2. Let us now shift A along the normal of $\overline{A_1A_2}$ so as to increase its distance from $\overline{A_1A_2}$. Since no diameter starts from A it can be attained that δ_A, i. e. the maximal distance of A from a vertex, is only slightly modified — while all diameters not starting from A

decrease according to our lemma 2. Thus we obtained a polygon each diameter of which may start only from A', which fact would contradict lemma 4.

7. To prove theorem 3 we construct a U_8[5]) the diameter of which is smaller than that of the regular U_8, i. e. smaller than $\left(\sin\frac{\pi}{8}\right)^{-1}$.

First we observe that the function

$$f(x)=\sqrt{1-\left(\frac{x-1}{2}\right)^2}+\sqrt{1-\frac{1}{4x^2}}+\sqrt{1-\left(\frac{x^2-x-1}{2x}\right)^2}-\sqrt{x^2-1}$$

has only one real zero in the interval $2\leqq x\leqq 3$ and this zero is smaller than $\left(\sin\frac{\pi}{8}\right)^{-1}$. Indeed, a numerical calculation yields

$$f(2)>0>f\left(\left(\sin\frac{\pi}{8}\right)^{-1}\right)>f(3)$$

while $\frac{d}{dx}f(x)<0$ for $2\leqq x\leqq 3$.

Denote this real zero of $f(x)$ by d and the positive quantities $\sqrt{1-\left(\frac{d-1}{2}\right)^2}$, $\sqrt{1-\frac{1}{4d^2}}$, $\sqrt{1-\left(\frac{d^2-d-1}{2d}\right)^2}$, $\sqrt{d^2-1}$ by p_1, p_2, p_3, p_4, respectively; we have evidently $p_1+p_2+p_3-p_4=0$.

Let us now consider the octagon determined by the points $P_1\left(x_1=\frac{1}{2}, y_1=0\right)$, $P_2\left(x_2=\frac{d}{2}, y_2=p_1\right)$, $P_3\left(x_3=\frac{d^2-1}{2d}, y_3=p_1+p_2\right)$, $P_4\left(x_4=\frac{1}{2}, y_4=p_1+p_2+p_3=p_4\right)$, $P_5(x_5=-x_4, y_5=y_4)$, $P_6(x_6=-x_3, y_6=y_3)$, $P_7(x_7=-x_2, y_7=y_2)$, $P_8(x_8=-x_1, y_8=y_1)$ on the (x, y) plane, which is obviously symmetric with respect to the y-axis.

Theorem 3 is proved if we show that this is a U_8 with diameter d.

At first it is easy to see that $\overline{P_iP_{i+1}}=1$ $(i=1, 2, \ldots, 8;\ P_9=P_1)$.

The convexity follows from the inequalities $y_1<y_2<y_3<y_4$, $x_1<x_2>x_3>x_4$ and from the fact that the projection of the side $\overline{P_2P_3}$ on the x-axis, i. e. $v_2=\sqrt{1-p_2^2}=\frac{1}{2d}$, is smaller than that of $\overline{P_3P_4}$, i. e. $v_3=\sqrt{1-p_3^2}=\sqrt{\frac{d^2-d-1}{2}}$.

Further we have

$$\overline{P_1P_5}=\overline{P_2P_6}=\overline{P_2P_7}=\overline{P_3P_7}=\overline{P_4P_8}=d,$$

and a simple calculation shows that the other diagonals are smaller than d, q. e. d.

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[5]) I am indebted for this example to my wife.