Fourier series with a sequence of positive coefficients.

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Let $f(\theta)$ be an integrable function of period 2π, with the Fourier series $\sum_{-\infty}^{\infty} c_n e^{in\theta}$. There are a number of theorems which indicate that if all c_n are nonnegative, then the magnitude of the c_n, and hence the behavior of $f(\theta)$, are controlled by the behavior of $f(\theta)$ near $\theta=0$. We mention two such results: if $c_n \geqq 0$ and $f(\theta)$ is bounded in a neighborhood of 0, then $\sum |c_n| < \infty$ and so $f(\theta)$ is continuous everywhere[1]); if $c_n \geqq 0$ and the pth derivative $f^{(p)}(\theta)$ exists at $\theta=0$, then $f^{(p-1)}(\theta)$ exists everywhere[2]). We shall generalize the latter theorem by assuming that the c_n are real and that the changes of sign in the sequence $\{c_n\}$ are not too frequent; our conclusion will be that $f(\theta)$ has at least a $(p-1)$th derivative everywhere if it has an integrable pth derivative in a sufficiently large neighborhood of $\theta=0$. Our theorem is, more precisely, as follows.

Theorem 1. *Let $f(\theta)$ be an integrable function of period 2π, with real Fourier coefficients c_n. Let k_n be the subsequence of integers at which a change of sign in the sequence c_n does not occur, and suppose that $|k_n - nB| < L$, where $B \geqq 1$ and L is a fixed positive number. Let p be a positive integer. If $f^{(p)}(\theta)$ exists and is integrable in the interval $(-\delta, \delta)$, where $\delta > \pi(1-B^{-1})$, then $f^{(p-1)}(\theta)$ exists for all θ. More generally, the same conclusion follows if, for some fixed positive A, we count c_n as "positive" if $c_n > -A|n|^{-p}$ and as "negative" if $c_n < A|n|^{-p}$, i. e., if we count no „change of sign" between c_n and c_{n+1} if either $c_n > -A|n|^{-p}$ and $c_{n+1} > -A|n+1|^{-p}$, or else $c_n < A|n|^{-p}$ and $c_{n+1} < A|n+1|^{-p}$.*

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1) R. E. A. C. PALEY: see, for example, G. H. HARDY and W. ROGOSINSKI, *Fourier Series* (Cambridge, 1944), p. 72.

2) R. FORTET, Calcul des moments d'une fonction de répartition à partir de sa caractéristique, *Bulletin des Sciences Math.*, (2) 68 (1944), pp. 117—131; H. CRAMÉR, *Mathematical Methods of Statistics* (Princeton, 1946), p. 90.

Proof. We have

$$2\pi c_n=\int_{-\delta}^{\delta} f(t)e^{-int}dt+\int_{\delta}^{\pi} f(t)e^{-int}dt+\int_{-\pi}^{-\delta} f(t)e^{-int}dt,$$

$$(-1)^n 2\pi c_n=(-1)^n\int_{-\delta}^{\delta} f(t)e^{-int}dt+\int_{\delta}^{\pi} f(t)e^{-in(t-\pi)}dt+\int_{-\pi}^{-\delta} f(t)e^{-in(t+\pi)}dt=$$

$$=(-1)^n\int_{-\delta}^{\delta} f(t)e^{-int}dt+\int_{\delta-\pi}^{0} f(t+\pi)e^{-int}dt+\int_{0}^{\pi-\delta} f(t-\pi)e^{-int}dt=$$

$$=(-1)^n\left\{\frac{f(\delta)e^{-in\delta}-f(-\delta)e^{in\delta}}{-in}+\dots+(-1)^{p-1}\frac{f^{(p-1)}(\delta)e^{-in\delta}-f^{(p-1)}(-\delta)e^{in\delta}}{(-in)^p}+\right.$$

$$\left.+\frac{(-1)^p}{(-in)^p}\int_{-\delta}^{\delta} f^{(p)}(t)e^{-int}dt\right\}+\int_{-(\pi-\delta)}^{\pi-\delta} g(t)e^{-int}dt,$$

where $g(t)=f(t-\pi\operatorname{sgn} t)$. Hence

$$n^p(-1)^n 2\pi c_n=\varphi(n)+\psi(n),$$

where

$$\varphi(n)=(-1)^n i^{-p}\int_{-\delta}^{\delta} f^{(p)}(t)e^{-int}dt,$$

and $\psi(n)$ is obtained by putting $z=n$ in

$$\psi(z)=\{iz^{p-1}[f(\delta)e^{iz(\pi-\delta)}-f(-\delta)e^{-iz(\pi-\delta)}]+\dots+$$

$$+(-1)^{p-1}i^p[f^{(p-1)}(\delta)e^{iz(\pi-\delta)}-f^{(p-1)}(-\delta)e^{-iz(\pi-\delta)}]+z^p\int_{-(\pi-\delta)}^{\pi-\delta} g(t)e^{-izt}dt\}.$$

Thus $\varphi(n)=O(1)$ as $n\to\infty$ and $\psi(z)$ is an entire function such that $|\psi(z)|\leqq\text{const.}\,e^{(\pi-\delta)|z|}$. Furthermore, $\psi(x)$ is real for real x, as follows from our assumption that all c_n are real.

Consider now an index k_n. If both $c_{k_n}>-A|k_n|^{-p}$ and $c_{k_n+1}>-A|k_n+1|^{-p}$, and (for example) if k_n is even and positive, we have $\psi(k_n)\geqq-2\pi A-\varphi(k_n)\geqq-C$, and $\psi(k_n+1)\leqq 2\pi A-\varphi(k_n+1)\leqq C$, where C is some constant; hence $|\psi(m_n)|\leqq C$ for some number m_n, where $k_n\leqq m_n\leqq k_n+1$. If $k_{n+1}=k_n+1$, then $c_{k_{n+1}}$ satisfies the same inequality as c_{k_n} and c_{k_n+1}, and we determine m_{n+1} similarly, $k_n+1\leqq m_{n+1}\leqq k_n+2$. If possible we select m_n and m_{n+1} so that $m_{n+1}-m_n\geqq\frac{1}{2}$. If this is not possible, we must have $\psi(x)>C$ for $k_n\leqq x\leqq k_n+\frac{1}{2}$ and for $k_n+\frac{3}{2}\leqq x\leqq k_n+2$, for otherwise we could choose either $k_n\leqq m_n\leqq k_n+\frac{1}{2}$ and $k_n+1\leqq m_{n+1}\leqq k_n+2$, or $k_n\leqq m_n\leqq k_n+1$ and $k_n+\frac{3}{2}\leqq m_{n+1}\leqq k_n+2$. Then since $\psi(k_n+1)\leqq C$, it follows that $\psi(x)$ has a minimum between $k_n+\frac{1}{2}$ and $k_n+\frac{3}{2}$, and so m_n and m_{n+1} can be chosen so that they are separated by a point q_n such that $\psi'(q_n)=0$. On the other hand, if $k_{n+1}>k_n+1$, we have $k_n\leqq m_n\leqq k_n+1$, $m_{n+1}\geqq k_n+2$, and so certainly $m_{n+1}-m_n>\frac{1}{2}$. Similar considerations apply for odd or negative n, or when the inequality satisfied by c_{k_n} is reversed.

Thus $\psi(z)$ satisfies $|\psi(z)|\leqq \text{const.}\, e^{(\pi-\delta)|z|}$, and $|\psi(m_n)|\leqq C$, where $|m_n-nB|\leqq L+1$ and either $|m_{n+1}-m_n|\geqq\frac{1}{2}$ or else $\psi'(q_n)=0$ with q_n between m_n and m_{n+1}. Now if $\delta>\pi(1-B^{-1})$, we have $\pi-\delta<\pi/B$ and by a result of DUFFIN and SCHAEFFER[3]), $\psi(x)$ is bounded on the whole real axis. (DUFFIN and SCHAEFFER require $|m_{n+1}-m_n|>\gamma>0$, but an analysis of their proof shows that the theorem remains valid without this restriction if $\psi'(x)$ vanishes between any two m_n's which differ by less than some fixed γ.) Since $\psi(x)$ is bounded, in particular $\psi(n)$ is bounded, and since $\varphi(n)$ is bounded, $n^p c_n$ is bounded. Hence $\sum|n^{p-1}c_n|^2$ converges and $f(\theta)$ has a $(p-1)$th derivative (belonging to L^2).

There is an analogous theorem for power series which can be proved in a similar way (it would be possible to formulate a general result including both theorems):

Theorem 2. *Let* $F(z)=\sum_{n=0}^{\infty}a_n z^n$ *for* $|z|<1$ *and suppose that for* $-\delta\leqq\theta\leqq\delta$ *and* $0<r<1$, *we have* $|F(re^{i\theta})|\leqq\omega(\theta)$, *where* $\omega(\theta)$ *is integrable; let* $F(z)$ *have a radial boundary function* $F(e^{i\theta})$ *for* $-\delta\leqq\theta\leqq\delta$, *such that* $F(e^{i\theta})$ *has an integrable* pth *derivative in* $-\delta\leqq\theta\leqq\delta$. *Let the* a_n *be real and let* k_n *be the subsequence of positive integers at which a change of sign in the sequence* $\{a_n\}$ *does not occur. If* $|k_n-nB|\leqq L$, *where* $B\geqq 1$ *and* L *is fixed, then* $a_n=O(n^{-p})$ *and consequently* $F(z)$ *has a radial boundary function, with at least* $p-1$ *derivatives, for all* θ.

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[3]) R. J. DUFFIN and A. C. SCHAEFFER, Power series with bounded coefficients, *American Journal of Math.*, **67** (1945), pp. 141—154.