Bilinear functionals over $C \times C$.

By MARSTON MORSE in Princeton, N. J.

§ 1. F. Riesz's theorem and the Fréchet generalization.

It is perhaps appropriate in this volume in honor of F. RIESZ and L. FEJÉR that a summary account be given of new and unpublished theorems on the representation and uses of bilinear functionals over the cartesian product $C \times C$. These results were recently obtained by the author and Dr. WILLIAM TRANSUE. The famous theorem of F. RIESZ on the representation of the most general functional f, linear over the Banach space C, as a Riemann-Stieltjes integral

$$f(x) = \int_0^1 x(s)\, dg(s) \qquad [x \in C] \tag{1.1}$$

(where g is a functional of bounded Jordan variation over the interval $[0,1]$) was followed by FRÉCHET's representation of the most general functional Φ bilinear (including continuous) over the cartesian product $C \times C$. FRÉCHET represents Φ by a repeated Riemann—Stieltjes integral

$$\Phi(x, y) = \int_0^1 x(s)\, d_s \int_0^1 y(t)\, d_t k(s, t) \qquad [x, y \in C]. \tag{1.2}$$

The distribution function k was required to have a special finite variation $P(E, k)$ (here termed an F-variation) over the unit interval $E = [0, 1] \times [0, 1]$ on which k was defined. Bearing in mind the celebrated contributions of FEJÉR to the theory of Fourier series I am happy to include in this account the innovation in the theory of the Pringsheim convergence of double Fourier series which our new theorems on the nature of the F-variation make possible. This report will be restricted to bilinear as distinguished from multilinear functionals, and to the F-variation over the 2-dimensional intervals I as distinguished from the F-variation over the corresponding n-dimensional interval $I^{(n)}$. The major part of our theorems have, however, been extended to the n-dimensional case (see MT 6). There remain outstanding difficulties which have been solved only for the case $n = 2$.

We shall recall the definition of $P[I, k]$ and extend this definition. Let E' and E'' respectively represent the interval $[0, 1]$ on the s and t-axes. We admit a partition π of $E = E' \times E''$ into subintervals defined by straight lines $[s = s_r]$ and $[t = t_n]$. The values s_r and t_n used to define π shall satisfy the conditions

$$0 = s_0 < s_1 < \dots < s_{r(\pi)} = 1, \tag{1.3}$$

$$0 = t_0 < t_1 < \dots < t_{n(\pi)} = 1. \tag{1.4}$$

For $r = 1, \dots, r(\pi)$; $n = 1, \dots, n(\pi)$ set

$$\Delta_{rn}(k) = k(s_r, t_n) - k(s_{r-1}, t_n) - k(s_r, t_{n-1}) + k(s_{r-1}, t_{n-1}). \tag{1.5}$$

Let e'_r be a constant, with $|e'_r| \leq 1$, associated with the rth interval of the partition (1.3) of E', and let e''_n with $|e''_n| \leq 1$ be similarly associated with the nth interval of the partition (1.4) of E''. We say that the set

$$[e'_1, \dots, e'_{r(\pi)};\ e''_1, \dots, e''_{n(\pi)}] = e \tag{1.6}$$

is *associated* with the partition π.

Then by definition

$$P(E, k) = \sup_{\pi, e} \sum_{r, n} e'_r e''_n \Delta_{rn}(k) \tag{1.7}$$

taking the sup over all admissible partitions π of E and associated sets e. We admit the possibility that $P(I, k) = +\infty$.

One immediately extends the definition of $P[I, k]$ over any closed subinterval $I = U \times V$ of E, restricting the partitioning values s_r to the interval U of the s-axis and the partitioning values t_r to the interval V of the t-axis. We also extend the definition of $P[I, k]$ to the case in which U and V may be open at either end point, both end points, or neither end point. In this case we set

$$P[I, k] = \sup_{J} P[J, k] \tag{1.8}$$

where J ranges over all closed subintervals of I. (MT 6, § 2.)

So defined $P[I, k]$ should be compared with the Vitali-variation $V[I, k]$. This may be defined as

$$V[I, k] = \sup_{\pi} \sum_{r, n} |\Delta_{rn}(k)|. \tag{1.9}$$

in case I is closed, and as $\sup_J V(J, k)$, taken as in (1.8), in case I is not closed. It is immediately obvious that

$$P[I, k] \leq V[I, k]. \tag{1.10}$$

It was known previously that functions k exist for which $P(I, k) < \infty$ and $V[I, k] = \infty$, but the example given was of a function k which vanished almost everywhere in E, and was inadequate for the purposes of our theory. (See Clarkson and Adams.) How much less restrictive numerical conditions on k

in terms of the F-variation are, than corresponding conditions in terms of the V-variation, is shown by the following theorem, established in MT 8, § 6.

Theorem 1.1. *Let I_{xy} be the interval $[0, x] \times [0, y]$, and let p be an arbitrary positive number. There exists a function k mapping E continuously into the axis of reals, vanishing on the boundary of E, absolutely continuous in the sense of Carathéodory over every closed subinterval of $(0, 1) \times (0, 1)$ and such that*

$$V[I_{xy}, k] = \infty, \quad P[I_{xy}, k] \leqq x^p y^p \tag{1.11}$$

for arbitrary positive x and y.

There are other important respects in which the F-variation differs from the V-variation. If I and J are 2-intervals intersecting in a common edge then

$$V[I \cup J, k] = V[I, k] + V[J, k] \tag{1.12}$$

while

$$P[I \cup J, k] \leqq P[I, k] + P[J, k] \tag{1.13}$$

with the equality in general not holding in (1.13). In addition the decomposition $k = P - N$ of k into two monotone functions P and N, possible when $V(I, k) < \infty$, is not in general possible when $P(I, k) < \infty$. In spite of these considerable differences the F-variation can be used with great advantage in place of the V-variation in many branches of analysis.

§ 2. Some basic properties of the F-variation.

The properties of the F-variation described in this section parallel in a remarkable way well known properties of the Jordan variation $T_a^b(g)$ over the interval $[a, b]$ of a function g with values $g(s)$ defined for $s \in [a, b]$. Assuming that $T_0^1(g) < \infty$ we list the known properties to which we shall give analogies for the F-variation.

I. The limits $g(s-)$ and $g(s+)$ exist for $s \in (0, 1]$ and $[0, 1)$ respectively.

II. The points at which g fails to be continuous are at most countably infinite.

III. If g^+ and g^- are functions defined by setting $g^+(s) = g(s+)$, $g^-(s) = g(s-)$ for $s \in (0, 1)$ and $g(0) = g^+(0) = g^-(0)$, $g(1) = g^+(1) = g^-(1)$ then

$$T_0^1(g^+) = T_0^1(g^-) \leqq T_0^1(g).$$

IV. Referring to the Riesz representation (1.1) we have

$$\sup_{x \in C} \frac{|f(x)|}{\|x\|} = T_0^1(g^+) = T_0^1(g^-) \quad (\|x\| \neq 0).$$

V. If $c < s < s' < 1$ then for fixed c and variable s and s' $\lim\limits_{s' \to c} T_s^{s'}(g) = 0$.

The analogies of these properties of the Jordan variation have been obtained in MT 1, 2, 6 and 10. We shall suppose that $P[E, k] < \infty$ and that on at least one section K of E on which $s=$ const., and on one section of E on which $t=$ const., the function $k|K$ defined by k over K has a finite Jordan variation. We say then that k is in $\widehat{F}(E)$. For $k \in \widehat{F}(E)$ the following holds:

I. *Let (a, b) be an arbitrary point in the (s, t) plane and let S_{ab} be any one of the four open quadrants into which the (s, t)-plane is divided by the lines $s=a$ and $t=b$. For fixed (a, b) and for $(s, t) \in S_{ab}$*

$$\lim_{(s,t)\to(a,b)} k(s,t) = \bar{k}^S(a,b) \qquad [(s,t)\in S_{ab}] \tag{2.1}$$

exists whenever S_{ab} intersects E. The four limits corresponding to the four quadrants may all be different. (MT 1, Theorem 5.1.)

II. *The points in E at which k fails to be continuous lie on a countable number of straight lines parallel to the coordinate axes.* (MT 1, Theorem 6.3.)

III. *Corresponding to any one of the variable quadrants S_{ab} of I taken with a fixed orientation, we shall define a function k^S over E. The detailed definition in case $s>a$ and $t>b$ in S_{ab} follows. Let $k^S(a,b)=\bar{k}^S(a,b)$ for $(a,b)\in(0,1)\times(0,1)$, with $k^S(a,b)=k(a,b)$ at each corner point of E. Let $k^S(0,t)=k(0,t+)$ and $k^S(1,t)=k(1,t+)$ for $t\in(0,1)$; and let $k^S(s,0)= = k(s+,0)$, $k^S(s,1)=k(s+,1)$ for $s\in(0,1)$. The remaining three functions k^S are similarly defined. Then $k^S(s,t)=k(s,t)$ at each point of continuity of k and for any two quadrant types S and S', of fixed orientation,*

$$P[E, k^S] = P[E, k^{S'}] \leqq P[E, k].$$

(MT 6, Theorem 8.2, § 6.)

IV. *Referring to the Fréchet representation* (1.2)

$$\sup_{x,y} \frac{|\varphi(x,y)|}{\|x\|\,\|y\|} = P[E, k^S] \qquad [0 \neq x\in C; 0 \neq y \in C]$$

where S is any one of the four quadrant types of fixed orientation. (MT 2, Theorem 12.1, MT 3, § 3.)

V. *Let S_{ab} be an open quadrant with vertex at $(a,b)\in E$. Let J be a variable 2-interval in $S_{ab}\cap E$. Then $P[J,k]\to 0$ as the maximum distance of the vertices of J from an edge of S_{ab} tends to zero.* (MT 10, Theorem 3.1.)

Property V has the following important application. If k is continuous over E and in $\widehat{F}(E)$, then $P[J,k]\to 0$ uniformly for arbitrary choice of $J\in E$ as the area of J tends to zero. (See MT 10, Corollary 3.5.)

We add the useful fact that when k is in $\widehat{F}(E)$, k is bounded and measurable over E. (See MT 4, § 2.)

§ 3. The Pringsheim convergence of double Fourier series.

On observing that the Dirichlet integral

$$(3.1)\qquad \frac{1}{\pi^2}\int_0^\pi\int_0^\pi f(s,t)\,\frac{\sin\left(m+\frac{1}{2}\right)s}{\sin\frac{s}{2}}\;\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\,ds\,dt$$

which gives the partial sum S_{mn} of the Fourier series for f, is really a special evaluation $\Phi(x,y)$ of a bilinear functional Φ with

$$x(s)=\frac{\sin\left(m+\frac{1}{2}\right)s}{\sin\frac{s}{2}},\qquad y(t)=\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}},$$

it is clear that the general theory of such functionals should be relevant to the convergence problem. Just as the second law of the mean is historically associated with the 1-dimensional Dirichlet integral, so here there is a generalized law of the mean which involves the F-variation of f, and is fundamental in treating the Dirichlet integral. As established in MT 10, Theorem 5.1, this law may be stated as follows.

Theorem 3.1. *Let f_1 and f_2 be integrable over* $[0, 1]$. *Let I the open interval $I=(0,1)\times(0,1)$. Suppose $g\in\widehat{F}(I)$ and that $g(0+,t)=0$ for $t\in(0,1)$ and $g(s,0+)=0$ for $s\in(0,1)$. Then*

$$(3.2)\qquad \left|\iint_I f_1(s)f_2(t)g(s,t)\,ds\,dt\right|\leqq\left|\int_c^1 f_1(s)\,ds\right|\left|\int_d^1 f_2(t)\,dt\right|\,P[I,g]$$

for some choice of c and d in $[0, 1]$.

With the aid of this theorem one can extend the 1-dimensional Jordan test as follows.

Theorem 3.2. *Let f be integrable over the interval $J=(0,2\pi)\times(0,2\pi)$, have the period 2π in each of its arguments, and be in $\widehat{F}(J)$. Then the Fourier series for f converges at each point (a, b) to the mean of the four open quadrant limits of f at (a,b). If f is continuous this convergence is uniform over J.* (MT 7, Theorem 1.)

As Theorem 1.1 indicates, the class of functions f which sastisfy this test is much larger than the class defined by HARDY in extending the Jordan test, using the V-variation of f. This theorem of course implies the Hardy theorem but not conversely.

We have weakened all of the classical two-dimensional tests (known to us) which use V-variations. These tests appear to include all the tests for Pringsheim convergence except the Tonelli test, and we prove that the Tonelli test is more restrictive than some of our "MT-tests". The tests so modified include those generalizing the Jordan, Dini, Young-Pollard, de la Vallée Poussin, Lebesgue and Gergen tests. (See MT 11.) We shall refer to the Dini test as typical.

If one is concerned with convergence at the origin one sets

$$4F(s,t)=f(s,t)+f(-s,t)+f(s,-t)+f(-s,-t)$$

and then writes $\varphi(s,t)=F(s,t)-c$. In the Young-Dini test the Fourier series of f converges to c if

$$\int_0^{\pi}\int_0^{\pi}\left|\frac{\varphi(s,t)}{st}\right| ds\,dt<\infty. \tag{3.3}$$

A much weaker version of this condition may be obtained as follows.

Let J be the interval $(0,\pi)\times(0,\pi)$. If g is defined over J and integrable over every closed subinterval of J, set

$$\bar{g}(u,v)=\int_{\pi}^{u}\int_{\pi}^{v} g(s,t)\,ds\,dt \qquad [(u,v)\in J].$$

The condition (3.3) may be written in the form

$$V[J,\overline{(\varphi/st)}]<\infty \qquad [\varphi=F-c] \tag{3.4}$$

and our weakened form of the Dini test is as follows.

Theorem 3.3. *If φ is integrable over $J=(0,\pi)\times(0,\pi)$, and if*

$$P[J,\overline{(\varphi/st)}]<\infty, \tag{3.5}$$

then the Fourier series of f converges to c at the origin.

This MT-Dini test is less restrictive than the Young-Dini test, as we show by an example. It is in fact so broad in its coverage that the class of functions φ which satisfy its conditions are included in none of the classical tests, not excepting the Gergen test (MT 11, § 11). We add the following

Theorem 3.4. *Each MT-test is less restrictive than the corresponding classical test.* (See Gergen for enumeration of classical tests.) *In particular the MT-test modifying the Gergen test is less restrictive than each classical test for Pringsheim convergence.* (See MT 1.1, Theorem 1.1.)

The last result is somewhat surprising since there is no proof known to us that the Gergen test in its original form is actually less restrictive than some of the other tests of Lebesgue type in the form given by Gergen. Our MT-Gergen test is, however, shown to be less restrictive than each of these tests.

Among the many theorems necessary in the calculus of Fréchet variations we shall state three which are typical.

Theorem 3.5. *Suppose g is integrable over every closed subinterval of $J=(0,1)\times(0,1)$ and that $P[J,\bar{g}]<\infty$. Let f_1 and f_2 be two functions in the Banach space M of functions essentially bounded over* $[0, 1]$. *Then the function $f_1 f_2 g$ with values $f_1(s)f_2(t)g(s,t)$ over J satisfies the condition*

$$P[J,\overline{(f_1 f_2 g)}]\leqq\|f_1\|\,\|f_2\|\,P[J,\bar{g}] \tag{3.6}$$

where $\|f_1\|$ and $\|f_2\|$ are norms of f_1 and f_2 in M. (MT 10, Theorem 6.2).

We point out that g may not be integrable over J so that it is possible that $V(J,\bar{g})=\infty$. In $\bar{g}$ we really have a Harnack integral with special properties. Under the conditions of the theorem one can show that $\bar{g}$ has a continuous extension over $\bar{J}$; we term this extension of $\bar{g}$ over $\bar{J}$ an FL-integral of g and develop its properties (MT 10, § 6). If $V[J,\bar{g}]=\infty$, then for g in the theorem

$$P[J,\bar{g}]<\infty, \qquad P[J,\overline{|g|}]=V[J,\bar{g}]=\infty.$$

Thus g may have an FL-integral $\bar{g}$ over $\bar{J}$ while $|g|$ is not FL-integrable.

We refer to a theorem of LEBESGUE. Let $x>0, y>0$ be positive infinitesimals. If g is in L over $J=(0,1)\times(0,1)$, according to Lebesgue

$$\int_0^{1-x}\int_0^{1-y} |\Delta_{xy}g(s,t)|\,dsdt=o(1), \tag{3.7}$$

where $\Delta_{xy}g(s,t)=g(s+x,t+y)-g(s,t+y)-g(s+x,t)+g(s,t)$.

If J^{xy} denotes the interval of integration in (3.7), one can write (3.7) in the form

$$V[J^{xy},\overline{\Delta_{xy}g}]=o(1). \tag{3.8}$$

Theorem 3.6. *If $\bar{g}$ is an FL-integral over $\bar{J}=[0,1]\times[0,1]$ then*

$$P[J^{xy},\overline{\Delta_{xy}g}]=o(1). \tag{3.9}$$

(MT 10, Theorem 8.1.)

As we have shown, (3.9) can hold without g being integrable over $\bar{J}$, or (3.7) holding. (MT 8, § 6.)

When g is integrable over $J=(0,1)\times(0,1)$, the mean $\widehat{g}/uv$, where

$$\widehat{g}(u,v)=\int_0^u ds\int_0^v g(s,t)\,dt \qquad [(u,v)\in J],$$

enters frequently, for example in the generalized de la Vallée Poussin test. The following theorem is then useful. See MT 10, Theorem 7.1 and Corollary 7.1.

Theorem 3.7. *If g is in L over $J=(0,1)\times(0,1)$ and $f_1(s)$ and $f_2(t)$ are positive, monotone decreasing and continuous over $(0,1)$, then*

$$P[J,f_1f_2\widehat{g}]\leq 4\,P[J,\overline{f_1f_2g}], \tag{3.10}$$

$$P[J,(\widehat{g}/st)]\leq P[J,g]. \tag{3.11}$$

In this theorem the right member of (3.10) can be finite while f_1f_2g is not integrable over J. (MT 8, § 6.)

§ 4. Variational theory.

The preceding results are a by-product of the studies of MORSE and TRANSUE in the variational theory of quadratic functionals. For the purposes of this variational theory one considers the product $A\times B$ of any two normed linear vector spaces A and B of which the products

$$L_2 \times L_2,\ L_2 \times L_1,\ C \times L_1,\ C \times L_2,\ C \times C$$

are important special cases. Let us suppose here that the spaces A and B are spaces of functions defined on the interval [0, 1]. Given any distribution function k mapping the interval $E = E' \times E''$ into the axis of reals, a variation $h(A, B, k)$ of k over E is defined generalizing the definition of $P[E, k]$. The Riesz—Fréchet representation theory is then extended and a variational theory and spectral theory initiated. This spectral theory is not a transformation theory but rather a direct critical point theory with the aspects characteristic of such a theory.

Bibliography.

Banach, S., *Théorie des opérations linéaires.* (Warsaw, 1932).

Clarkson, J. A. and Adams, C. R., On definitions of bounded variation for functions of two variables, *Transactions American Math. Society,* **35** (1933), pp. 824–854.

Fréchet, M., Sur les fonctionnelles bilinéaires, *Ibidem,* **16** (1915), pp. 215—234.

Gergen, J. J., Convergence criteria for double Fourier series, *Ibidem,* **35** (1933), pp. 29–63.

Hardy, G. H., On double Fourier series, and especially those which represent the double zeta-function with real and incommensurable parameters, *Quarterly Journal of Math.,* **37** (1906), pp. 53–79.

Morse, M. and Transue, W., 1. Functionals of bounded Fréchet variation, *Canadian Journal of Math.,* **1** (1949), pp. 153—165. — 2. Functionals F bilinear over the product $A \times B$ of two p-normed vector spaces. I. The representation of F, *Annals of Math.,* **50** (1949). — 3. Functionals F bilinear over the product $A \times B$ of two p-normed vector spaces. II. Admissible spaces A, *Annals of Math.,* **51** (1950). — 4. Integral representations of bilinear functionals, *Proceedings National Academy of Sciences U. S. A.,* **35** (1949), pp. 136—143. — 5. A characterization of the bilinear sums associated with the classical second variation, *Annali di Matematica,* **29** (1950). — 6. The Fréchet variation in the small, sector limits, and left decompositions (To be published). — 7. The Fréchet variation and a generalization for multiple Fourier series of the Jordan test. (To be published). — 8. Norms of distribution functions associated with bilinear functionals. (To be published). — 9. The Fréchet variation and the convergence of multiple Fourier series, *Proceedings National Academy of Sciences U. S. A.,* **35** (1949), pp. 395—399. — 10. Calculus for Fréchet variations. (To be published). — 11. The Fréchet variation and Pringsheim convergence of double Fourier series. (To be published).

Riesz, F., Sur les opérations fonctionnelles linéaires, *Comptes Rendus de l'Académie des Sciences Paris,* **149** (1909), pp. 974—977.

Institute for Advanced Study.

(Received July 29, 1949.)