# Resolutions of the identity for commutative $B^*$-algebras of operators.

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In this note we give a generalization of the well-known spectral formula $f(T)=\int f(\lambda)\,dE_\lambda$. The theorem we prove (Theorem 3) is based upon a theorem of GELFAND—NEUMARK[1]) and upon a generalization of F. RIESZ's[2]) classical result concerning the linear functionals on the space $C$. For the sake of completeness these two theorems (Theorems 1 and 2) are stated explicitly but without proof. The general idea of the proof was discovered by F. RIESZ[3]).

## § 1. Preliminary concepts.

1. 1. **The Riesz representation theorem.** In what follows we shall be concerned with a compact (= bicompact) Hausdorff space $\Lambda$ and the family $C(\Lambda)$ of complex valued continuous functions defined on $\Lambda$. The space $C(\Lambda)$ is a complex Banach space under the norm $|f|=\sup_\lambda |f(\lambda)|$ and as such determines a conjugate space $C^*(\Lambda)$, namely the space of all continuous linear maps of $C(\Lambda)$ into the complex number system. Besides being linear spaces $C(\Lambda)$ and $C^*(\Lambda)$ are partially ordered via the following definitions: for $f\in C(\Lambda)$ we say that $f\geqq 0$ if $f(\lambda)\geqq 0$, $\lambda\in\Lambda$; and for $x^*\in C^*(\Lambda)$ we say that $x^*\geqq 0$ if $x^*f\geqq 0$ for every $f\geqq 0$. It will be necessary in what follows to have a representation for the space $C^*(\Lambda)$. A representation for $C^*(\Lambda)$ was originally found, in the case where $\Lambda$ is an interval of real numbers, by F. RIESZ. RIESZ's theorem has been successively generalized by RADON, SAKS, VON

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[1]) I. GELFAND—M. NEUMARK, On the embedding of normed rings into the ring of operators in Hilbert space, *Mat. Sbornik*, N. S., **12** (1943), pp. 197—213. See also the very elegant proof of R. ARENS, On a theorem of Gelfand—Neumark, *Proceedings National Academy of Sciences*, **32** (1946), pp. 237—239.

[2]) F. RIESZ, Sur les opérations fonctionnelles linéaires, *Comptes Rendus de l'Académie des Sciences Paris*, **149** (1909), pp. 974—977.

[3]) F. RIESZ, Über quadratische Formen von unendlich vielen Veränderlichen, *Göttinger Nachrichten*, 1910, pp. 190—195.

NEUMANN, KAKUTANI and others. In order to state here the form of the generalized Riesz theorem that we shall need, it is necessary to introduce the notion of a regular measure on $\Lambda$. A function $\mu$ is called a *regular measure* on $\Lambda$ if it is a countably additive complex valued set function defined on the field $B$ of all Borel subsets of $\Lambda$ and in case it also has the following property: for every Borel set $e \subset \Lambda$ and every $\varepsilon > 0$ there is a closed set $a \subset e$ and an open set $b \supset e$ such that for every $e' \in B$ with $a \subset e' \subset b$ we have $|\mu(e) - \mu(e')| < \varepsilon$. It is not difficult to see that the regular measures on $\Lambda$ form a complex Banach space under the norm $|\mu| =$ total variation of $\mu(e)$, $e \in B$. For brevity we shall denote this complex Banach space of regular measures on $\Lambda$ by the symbol $R(\Lambda)$. The space $R(\Lambda)$ is partially ordered by the definition $\mu \geqq 0$ if and only if $\mu(e) \geqq 0$, $e \in B$. With these notions in mind then the general form of the Riesz representation theorem may be stated as follows.

Theorem 1. *For every $x^* \in C^*(\Lambda)$ there is a uniquely determined point $\mu \in R(\Lambda)$ for which*

$$x^* f = \int_\Lambda f(\lambda)\, d\mu(\lambda), \qquad f \in C(\Lambda).$$

*This correspondence $x^* \to \mu$ is an order preserving, isometric isomorphism between the spaces $C^*(\Lambda)$ and $R(\Lambda)$.*

A proof of the theorem, in essentially this form, may be found in a paper of KAKUTANI[4]). KAKUTANI considers only the real continuous functions on $\Lambda$ and represents the positive functionals. A proof of the theorem as we have stated it here may be based upon KAKUTANI's work and presents no real difficulty.

1.2. **The Gelfand—Neumark representation theorem.** The particular compact Hausdorff space $\Lambda$ to which Theorem 1 is to be applied is the space of maximal ideals of a certain commutative $B^*$-algebra. By a $B$-algebra we shall mean a complex Banach space $\mathfrak{A}$ with the properties

($\alpha$) There is a binary operation defined in $\mathfrak{A}$ with the properties

$$x(\lambda y + \mu z) = \lambda xy + \mu xz, \quad (\lambda y + \mu z)x = \lambda yx + \mu zx, \quad x(yz) = (xy)z,$$

where $\lambda, \mu$ are complex numbers and $x, y, z \in \mathfrak{A}$.

($\beta$) There is a unit $e \in \mathfrak{A}$ with the property that $ex = x$, $x \in \mathfrak{A}$.

($\gamma$) $|xy| \leqq |x| . |y|$ for $x, y \in \mathfrak{A}$; $|e| = 1$.

A $B$-algebra $\mathfrak{A}$ is called a $B^*$-algebra in case there is a unary operation $*$ in $\mathfrak{A}$ with the properties

($\delta$) $(x^*)^* = x$, $(\alpha x)^* = \bar{\alpha} x^*$, $(xy)^* = y^* x^*$, $(x+y)^* = x^* + y^*$, $|x^* x| = |x|^2$.

A $B$-algebra $\mathfrak{A}$ is called commutative in case $xy = yx$, $x, y \in \mathfrak{A}$. If $\lambda$ is a

[4]) S. KAKUTANI, A concrete representation of abstract ($M$)-spaces, *Annals of Math.*, **42** (1941), pp. 994–1024; in particular Theorem 9, p. 1009.

maximal ideal in a commutative $B$-algebra, GELFAND[5]) has shown that the quotient algebra $\mathfrak{A}/\lambda$ is the complex number system. The complex number corresponding to an element $x \in \mathfrak{A}$ under the natural homomorphism $\mathfrak{A} \to \mathfrak{A}/\lambda$ will be denoted by $x(\lambda)$. If in the set $\Lambda$ of all maximal ideals of $\mathfrak{A}$ we define neighborhoods of a point $\lambda_0 \in \Lambda$ to be all sets of the form

$$N(\lambda_0) = [\lambda:\ \lambda \in \Lambda,\ |x_i(\lambda) - x_i(\lambda_0)| < \varepsilon; i = 1, \ldots, n],$$

where $x_i$ $(i = 1, \ldots, n)$ are arbitrary elements of $\mathfrak{A}$ and $\varepsilon$ is an arbitrary positive number, then, as GELFAND has shown, $\Lambda$ becomes a compact Hausdorff space. The functions $x(\lambda)$ are continuous on $\Lambda$ and $\sup|x(\lambda)| \leq |x|$. In case the commutative $B$-algebra is also a $B^*$-algebra, much more can be said; namely,

Theorem 2. (GELFAND—NEUMARK). *Let $\Lambda$ be the compact Hausdorff space of maximal ideals of the commutative $B^*$-algebra $\mathfrak{A}$. Then $x^*(\lambda) = \overline{x(\lambda)}$ (the complex conjugate of $x(\lambda)$). Furthermore, the map $x \to x(\lambda)$ of $\mathfrak{A}$ into $C(\Lambda)$ is an isometric isomorphism of $\mathfrak{A}$ onto $C(\Lambda)$.*

## § 2. Resolutions of the identity for commutative $B^*$-algebras of operators in Hilbert spaces.

2. 1. **The algebras $\mathfrak{A}$ and $\mathfrak{A}(H)$.** Let $H$ be a Hilbert space (not necessarily separable) and let $\mathfrak{A}(H)$ be the algebra of all continuous linear transformations in $H$. For $T \in \mathfrak{A}(H)$ let $T^*$ be the adjoint of $T$ and let $|T| = \sup_{|x|=1} |Tx|$. Then $\mathfrak{A}(H)$ is a $B^*$-algebra with unit $I$. In what follows the symbol $\mathfrak{A}$ will be used for any closed commutative $B^*$ subalgebra of $\mathfrak{A}(H)$ which contains the unit $I$. The symbol $\Lambda$ will be used for the compact Hausdorff space of maximal ideals in $\mathfrak{A}$. For a function $f \in C(\Lambda)$ the symbol $\bar{f}$ will denote the function $\bar{f}(\lambda) = \overline{f(\lambda)}$. Thus, according to Theorem 2, there is an isometric isomorphism, $f \leftrightarrow T(f)$, between the algebras $C(\Lambda)$ and $\mathfrak{A}$, and this isomorphism has the property that $T(f)^* = T(\bar{f})$.

2. 2. **The resolution of the identity corresponding to the isomorphism $\mathbf{f} \leftrightarrow \mathbf{T(f)}$.** Let $f$, $T(f)$ be corresponding elements of $C(\Lambda)$ and $\mathfrak{A}$ under some isometric isomorphism. The symbol $B$ will stand for the Borel subsets of $\Lambda$.

Theorem 3. *For $e \in B$ there is a uniquely determined $E_e \in \mathfrak{A}(H)$ with the properties*

(i) *For $x \in H$ the function $E_e x$ is countably additive on $B$.*

(ii) *For every pair $x, y \in H$, the scalar product $(E_e x, y)$ is a regular countably additive set function on $B$ whose total variation is at most $|x|\,|y|$.*

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[5]) I. GELFAND, Normierte Ringe, *Mat. Sbornik*, N. S., **9** (1941), pp. 1—24.

(iii) *For every* $T\in\mathfrak{A}$ *and* $e, e_1, e_2\in B$

$$E_e T = T E_e,\ E_{e_1}E_{e_2} = E_{e_2}E_{e_1},\ E_e^2 = E_e,\ E_e^* = E_e.$$

(iv) $(T(f)x, y) = \int_\Lambda f(\lambda)\,d(E_\lambda x, y),\ x, y\in H,\ f\in C(\Lambda).$ [6])

The number $(T(f)x, y)$ is clearly linear in $f$ and since $|(T(f)x, y)|\leqq \leqq |T(f)||x||y| = |f||x||y|$ there is, by Theorem 1, a uniquely determined regular measure $\mu(e, x, y)$ on $\Lambda$ with

(a) $$(T(f)\,x, y) = \int_\Lambda f(\lambda)\,d\mu(\lambda, x, y),\quad f\in C(\Lambda),$$

(b) $$|\mu(e, x, y|\leqq \operatorname*{var}_{e\in B}\mu(e, x, y)\leqq |x|\,|y|,\quad e\in B.$$

Since $(T(f)\,\alpha x, y) = \alpha(T(f)\,x, y)$ we have from (a) $\int_\Lambda f(\lambda)\,d\mu(\lambda, \alpha x, y) = = \int_\Lambda f(\lambda)\,d\alpha\mu(\lambda, x, y)$ and since the regular measure is uniquely determined by the functional it is seen that $\mu(e, \alpha x, y) = \alpha\mu(e, x, y)$. Similarly it may be shown that $\mu(e, x, y)$ is bilinear in $x, y$. It is also Hermitian symmetric for if $f$ is real the operator $T(f) = T(\bar f) = T(f)^*$ is self-adjoint and $[T(f)\,x, y] = = \overline{(T(f)\,y, x)}$. Thus by (a), $\int_\Lambda f(\lambda)\,d\mu(\lambda, x, y) = \int_\Lambda f(\lambda)\,\overline{d\mu(\lambda, y, x)}$ for $f$ real, and again by the uniqueness argument $\mu(e, x, y) = \overline{\mu(e, y, x)}$. Hence $\mu(e, x, y)$ is an Hermitean symmetric bilinear form of norm (see (b)) at most unity. It follows from the well-known elementary theorem which gives the representation of such forms that there is a self-adjoint operator $E_e$ of norm $|E_e|\leqq 1$ with $\mu(e, x, y) = (E_e x, y)$. This fact when combined with (a) and (b) proves (ii) and (iv). To see that $TE_e = E_e T$ let $f, g$ be arbitrary elements of $C(\Lambda)$ and since $T(f)$, $T(g)$ commute we have $(T(f)\,T(g)\,x, y) = (T(f)\,x,\ T(g)^* y)$. Equation (iv) then gives $\int f(\lambda)\,d(E_\lambda T(g)\,x, y) = \int f(\lambda)\,d\,(E_\lambda x,\ T(g)^* y)$ and since the regular measure is uniquely determined by the integral,

$$(E_e T(g)\,x, y) = (E_e x,\ T(g)^* y) = (T(g)\,E_e x, y),\ \text{i. e.,}\ E_e T(g) = T(g)\,E_e.$$

We now proceed to a proof of the relation $E_{e_1}E_{e_2} = E_{e_1 e_2}$. For this we shall need the following lemmas.

Lemma 1. *If* $f(\lambda)\geqq 0$ *is continuous and* $\mu(e)$ *is a non-negative regular measure on* $\Lambda$ *and if* C *is a closed set in* $\Lambda$ *then*

$$\text{g. l. b.}\int_\Lambda f(\lambda)\,g(\lambda)\,d\mu(\lambda) = \int_C f(\lambda)\,d\mu(\lambda),$$

[6]) If the algebra is separable, or if the Hilbert space itself is separable, the theorem is a corollary of known results. See J. v. Neumann, Zur Algebra der Funktionaloperationen und Theorie der normalen Operatoren, *Math. Annalen*, **102** (1929), pp. 370–427, Satz 10, or Béla v. Sz. Nagy, *Spektraldarstellung linearer Transformationen des Hilbertschen Raumes* (Berlin, 1942), pp. 66–67.

*where the g. l. b. is taken over* $g \in C(\Lambda)$ *for which* $g(\lambda) \geqq \varphi_C(\lambda)$, *the characteristic function of* $C$.

We shall first prove the lemma in case $f(\lambda) \equiv 1$. Let $G$ be an open set containing $C$. Since $\Lambda$ is a normal topological space, there is a $g \in C(\Lambda)$ with

$$0 \leqq g(\lambda) \leqq 1; \quad g(\lambda) = 1 \text{ for } \lambda \in C; \quad g(\lambda) = 0 \text{ for } \lambda \bar{\in} G.$$

Thus

$$\mu(G) \geqq \int_\Lambda g(\lambda)\, d\mu(\lambda) \geqq \mu(C)$$

and since $\mu$ is regular, $\mu(G)$ differs from $\mu(C)$ by as little as we please. This proves the lemma in case $f(\lambda) \equiv 1$. For the general case we let $\mu_1(e) = \int_e f(\lambda)\, d\mu(\lambda)$ and observe that $\mu_1$ is a non-negative regular measure which satisfies the equation

(c) $$\int_\Lambda g(\lambda) f(\lambda)\, d\mu(\lambda) = \int_\Lambda g(\lambda)\, d\mu_1(\lambda)$$

providing $g$ is the characteristic function of a Borel set in $\Lambda$. Since an arbitrary $g \in C(\Lambda)$ may be uniformly approximated by linear combinations of characteristic functions of Borel sets, it is seen that (c) holds for every continuous function. The desired conclusion follows then by an application of the case $f(\lambda) \equiv 1$, already proved, to equation (c).

Lemma 2. $(E_e x, x) \geqq 0$ *for* $e \in B$, $x \in H$.

In view of Theorem 1 it will suffice to show that $\int f(\lambda) d(E_\lambda x, x) \geqq 0$ for every continuous positive function $f$. For such an $f$, $f^{\frac{1}{2}}(\lambda)$ is real and hence $T(f^{\frac{1}{2}})$ is self-adjoint. Thus $0 \leqq |T(f^{\frac{1}{2}})x|^2 = (T(f^{\frac{1}{2}})x, T(f^{\frac{1}{2}})x) = (T(f)x, x) = \int_\Lambda f(\lambda) d(E_\lambda x, x)$, q. e. d.

Now let $f, g$ be non-negative and continuous. Then by Lemma 2 we have

$$0 \leqq \int f(\lambda) g(\lambda)\, d(E_\lambda x, x) = (T(fg)x, x) = (T(f)x, T(g)x) = \int f(\lambda)\, d(E_\lambda x, T(g)x).$$

Since $(E_e x, T(g)x) = (T(g)x, E_e x) = \int g(\mu)\, d(E_\mu x, E_e x)$, the above equation may be written as

(d) $$0 \leqq \int f(\lambda) g(\lambda)\, d(E_\lambda x, x) = \int f(\lambda)\, d \int g(\mu)\, d(E_\mu x, E_\lambda x).$$

Since $f$ and $g$ are arbitrary non-negative continuous functions, we see from Theorem 1 first that $\int g(\mu)\, d(E_\mu x, E_e x) \geqq 0$ and then that

(e) $$(E_{e_1} x, E_e x) \geqq 0 \text{ for } e, e_1 \in B.$$

Let $C$ be a closed set, let $\varphi$ be the characteristic function of $C$ and let $\mu(e)=(E_{eC}x, x)$. Then from (e)

$$\int g(\mu)\, d(E_\mu x, E_e x) \geqq (E_C x, E_e x), \quad g \geqq \varphi,$$

and so by (d)

$$\int f(\lambda)\, g(\lambda)\, d(E_\lambda x, x) \geqq \int f(\lambda)\, d(E_C x, E_\lambda x), \quad f \geqq 0, \ g \geqq \varphi.$$

By Lemma 1, then

$$\int f(\lambda)\, d\mu(\lambda) = \int_C f(\lambda)\, d(E_\lambda x, x) \geqq \int f(\lambda)\, d(E_C x, E_\lambda x)$$

which gives

$$\text{(f)} \qquad (E_{eC}x, x) = \mu(e) \geqq (E_C x, E_e x) = (E_e E_C x, x).$$

Now let $g \geqq \varphi$, then by (d)

$$\int f(\lambda)\, d\mu(\lambda) = \int_C f(\lambda)\, d(E_\lambda x, x) \leqq \int f(\lambda)\, g(\lambda)\, d(E_\lambda x, x) =$$

$$= \int f(\lambda)\, d \int g(\mu)\, d(E_\mu x, E_\lambda x),$$

and since the measures are regular,

$$\mu(e) = (E_{eC}x, x) \leqq \int g(\mu)\, d(E_\mu x, E_e x); \quad e \in B.$$

An application of Lemma 1 gives $(E_{eC}x, x) \leqq (E_C x, E_e x) = (E_e E_C x, x)$, which when combined with (f) yields $(E_{eC}x, x) = (E_e E_C x, x)$. Assume for the moment that $E_e$ and $E_C$ commute, then the operator $A = E_{eC} - E_e E_C$ is self-adjoint and since $(Ax, x) = 0$ it follows that $A = 0$. For fixed $e \in B$ the two regular measures $(E_{ea}x, y)$, $(E_e E_a x, y)$ coincide for closed sets $a$. They must therefore coincide everywhere on $B$. Thus $E_e E_a = E_{ea}$, $E_e^2 = E_e$. It remains to be shown that $E_{e_1} E_e = E_e E_{e_1}$. But this follows immediately from the fact that $E_e$ is self-adjoint and commutes with $T(f)$. For from the equation $(T(f) E_{e_1} x, y) = (T(f)x, E_{e_1} y)$ we get $\int f(\lambda)\, d(E_\lambda E_{e_1} x, y) = \int f(\lambda)\, d(E_\lambda x, E_{e_1} y)$ and thus $(E_e E_{e_1} x, y) = (E_e x, E_{e_1} y) = (E_{e_1} E_e x, y)$. Only statement (1) remains to be proved. Let $e_n$ be disjoint, $e_n \in B$, $e = \bigcup_n e_n$ and $a_n = e - \bigcup_{m=1}^{n} e_m$. Then we must show that $E_{a_n} x \to 0$. But $|E_{a_n} x|^2 = (E_{a_n} x, E_{a_n} x) = (E_{a_n} x, x) \to 0$ by (ii). This completes the proof of the theorem.

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