The location of critical points of harmonic functions.

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There has recently been developed an extensive theory of the location of the critical points of harmonic functions, namely the study of the geometry of regions containing some or all critical points of a given harmonic function; compare a forthcoming volume by the present writer in the Colloquium Series of the American Mathematical Society. The most useful method in establishing that theory is the interpretation of the given harmonic function as the potential due to a suitable distribution of matter, and detailed study of the corresponding field of force. The latter is usually set up by means of GREEN's theorem or CAUCHY's integral. It is to be expected that the classical results of F. RIESZ on the representation of superharmonic functions[1]) would also be of significance here; indeed, it is the purpose of the present note to indicate the great power of his results in this study of critical points when superharmonic functions are involved.

In the plane of $z=x+iy$, a function $u(z)\equiv u(x,y)$ is *harmonic* at a finite point z if throughout a neighborhood of that point the function is continuous together with its first and second partial derivatives, and satisfies LAPLACE's equation. A function $u(z)$ is harmonic *in a region* if it is harmonic at every point of that region. A finite point (x_0, y_0) is a *critical point* of a function $u(x,y)$ harmonic at (x_0, y_0) if the two first partial derivatives of $u(x,y)$ vanish there. A critical point remains a critical point under one-to-one conformal transformation.

We state for reference only a small part of the results of F. RIESZ:

If the function $u(z)$ is superharmonic in the (regular) region R, and if there exists a function harmonic in R inferior to $u(z)$ throughout R, then $u(z)$ can be represented in R in the form

$$(1)\qquad u(z)\equiv\int_R g(z,t,R)\,d\mu(e)+h(z),\quad d\mu(e)\geqq 0,$$

where $g(z,t,R)$ is Green's function for R with pole in z, where as t varies

[1]) F. RIESZ, Sur les fonctions subharmoniques et leur rapport à la théorie du potentiel. II, *Acta Math.*, **54** (1930), pp. 321–360.

over R the integral is taken with respect to a suitably chosen positive additive set function $\mu(e)$ defined for t on all open sets e whose closures lie in R, and where $h(z)$ is harmonic in R.

In the particular region R_1: $|z|<1$ we shall consider the NE lines, that is to say, the lines of non-euclidean (hyperbolic) geometry, namely the arcs in R_1 of circles and straight lines orthogonal to the circumference $|z|=1$. We define NE lines in an arbitrary region that can be mapped one-to-one and conformally onto R_1 as the images of the NE lines in R_1, and shall establish

Theorem 1. *Let R be the interior of a Jordan curve C, and let B be a closed set in R which together with C bounds a subregion R' of R. Let $u(z)$ be harmonic in R', superharmonic in R, continuous in the closed neighborhood of C in $R+C$, and zero on C. Then all critical points of $u(z)$ in R' lie in the smallest NE convex set Π of R containing B.*

We map R onto the interior of the unit circle $|z|=1$ so that an arbitrary preassigned point of R' not in Π is transformed into the origin 0. We preserve the original notation, and choose an arbitrary NE line through 0 wholly in R' as the axis of reals, with B in the upper half-plane. It remains to show merely that 0 is not a critical point of $u(z)$. Equation (1) is a consequence of RIESZ's theorem. Since both $u(z)$ and $h(z)$ are harmonic in R', it follows from (1) that the integral is harmonic in R', and follows further that we have $d\mu=0$ in R'. Consequently the integral may be taken not over R but over an arbitrary open set in R containing $R-R'$. The function $g(z,t,R)$ considered as a function of z for t in $R-R'$ can be extended harmonically by reflection across C; the values at points mutually inverse with respect to C are the negatives of each other, so the integral represents a function of z harmonic in an annular region containing C, a function vanishing on C itself. It follows from (1) that $h(z)$ is continuous on $R+C$ and zero on C, hence identically zero in R. Thus equation (1) reduces to

$$u(z)=\int_{R-R'} g(z,t,R)\,d\mu(e), \quad z \text{ in } R; \tag{2}$$

more accurately, the integral is to be taken over an arbitrary open subset of R containing $R-R'$.

We can write explicitly for z in R

$$g(z,t,R)\equiv\log|z-1/\bar{t}|-\log|z-t|+\log|t|,$$

so if we set $f(z)\equiv u(z)+iv(z)$, where $v(z)$ is conjugate to $u(z)$ in R', we have for z in R for a suitable determination of the multiple-valued functions involved,

$$f(z)=\int_{R-R'}[\log(z-1/\bar{t})-\log(z-t)+\log|t|]\,d\mu+\text{const},$$

$$f'(z) = \int_{R-R'} \frac{d\mu}{z-1/\bar{t}} - \int_{R-R'} \frac{d\mu}{z-t}, \quad z \text{ in } R'; \tag{3}$$

of course $f'(z)$ is single-valued in R'.

The conjugate of the number $\mu/(z-\alpha)$ where μ is real, represents generically the force at the point z due to a particle of mass μ at the point α, where the particle repels with a force equal to the mass divided by the distance. Thus the conjugate of the second member of (3) represents the force at z due to a spread of positive mass at the (finite) points $1/\bar{t}$ and numerically equal negative mass at the points t. The force at the particular point $z=0$ due to the mass μ (>0) at $1/\bar{t}$ and the mass $-\mu$ at t is

$$\frac{\mu}{-1/t} - \frac{\mu}{-\bar{t}} = \mu\left(\frac{1}{\bar{t}} - t\right),$$

which when t lies in R in the upper half-plane is a vector with a non-vanishing component vertically upward. Thus it follows from (3) that the total force at $z=0$ has a non-vanishing component vertically upward, so we have $f'(0) \neq 0$ and hence 0 is not a critical point of $u(z)$; Theorem 1 is established.

As a special case of Theorem 1 we have

Theorem 2. *Let R be the interior of a Jordan curve C, and let B be a closed set in R which together with C bounds a subregion R' of R. Let $u(z)$ be harmonic in R', continuous in $R'+B+C$, zero on C, and unity on B. Then all critical points of $u(z)$ in R' lie in the smallest NE convex set of R containing B.*

Extend the definition of $u(z)$ if necessary so that $u(z)$ is defined and equal to unity at every point of $R-(R'+B)$. Then $u(z)$ as thus extended is superharmonic in R, for $u(z)$ is locally superharmonic; it is sufficient to note that $u(z)$ is continuous in R, is harmonic at every point of R' and at every (interior) point of $R-(R'+B)$, and at an arbitrary point P of B is not less than the average of the values of $u(z)$ over any sufficiently small circumference whose center is P. Theorem 2 follows.

The contrast between Theorems 1 and 2 indicates the advantage of the methods of F. RIESZ, for Theorem 2 can be proved[2]) without the use of (1) and (2) by establishing an equation similar to (3) where the integral is a line integral taken over an auxiliary variable point set B', chosen as a level locus of $u(z)$ in R', and consisting of mutually exterior analytic Jordan curves. Theorem 1 obviously applies to functions $u(z)$ much more general than those of Theorem 2, indeed applies to any linear combination with

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[2]) J. L. WALSH, Note on the location of zeros of the derivative of a rational function whose zeros and poles are symmetric in a circle, *Bulletin American Math. Society*, **45** (1939), pp. 462–470.

positive constant coefficients of functions of the type of Theorem 2, each such function defined as above throughout R.

In Theorem 2 we may choose $u(z)$ a constant multiple of $-\log|R(z)|$, with

$$R(z)=\prod_{1}^{n}\frac{z-\alpha_k}{1-\bar{\alpha}_k z}, \qquad |\alpha_k|<1,$$

and where B is a level locus of $u(z)$; the critical points of $u(z)$ are precisely the critical points of $R(z)$, with the omission of the multiple zeros of $R(z)$. Theorem 2 then implies that *all critical points of $R(z)$ in R lie in the smallest NE convex set in R containing the points α_k;* this theorem is a NE analogue of the classical theorem of LUCAS regarding the zeros of the derivative of a polynomial; well-known beautiful applications of LUCAS's theorem have been made by L. FEJÉR.

As a further illustration of the method of F. RIESZ we prove

Theorem 3. *Let Π_1 and Π_2 be the upper and lower half-planes, let R be the region $|z|<1$, and let a subregion R' of R be bounded by the unit circle C and a closed set B in R not intersecting the axis of reals A. Let the function $u(x,y)$ be harmonic in R', continuous in the closed neighborhood of C in $R+C$, zero on C, superharmonic in $R\cdot\Pi_1$, and subharmonic in $R\cdot\Pi_2$. Then 0 is not a critical point of $u(x,y)$.*

We assume, as we may do with no loss of generality, that R' is symmetric in A and that we have $u(x,y)\equiv-u(x,-y)$ in R'; for we need merely replace a given $u(x,y)$ by the function $u_1(x,y)\equiv u(x,y)-u(x,-y)$ to obtain these conditions. Moreover we shall prove $\partial u_1(0,0)/\partial y\neq 0$, which implies $\frac{1}{2}\partial u_1(0,0)/\partial y=\partial u(0,0)/\partial y\neq 0$. We revert to the original notation, and note in particular that the condition $u(x,y)\equiv-u(x,-y)$ implies $u(x,0)\equiv 0$.

The function $u(z)$ is superior to the function zero in the region $R\cdot\Pi_1$, so by F. RIESZ's theorem we have for z in $R\cdot\Pi_1$

$$u(z)=\int_{R\cdot\Pi_1} g(z,t,R\cdot\Pi_1)\,d\mu+h_1(z), \quad d\mu\geqq 0, \tag{4}$$

where $h_1(z)$ is harmonic in $R\cdot\Pi_1$. Throughout the region $R'\cdot\Pi_1$ the functions $u(z)$ and $h_1(z)$ are harmonic and $d\mu$ is zero, so the integral can be taken over an arbitrary open set in $R\cdot\Pi_1$ containing the set $(R-R')\cdot\Pi_1$. The function $g(z,t,R\cdot\Pi_1)$ considered as a function of z can be extended harmonically across A, then across C, so the integral in (4) represents a function continuous on A and C and vanishing there. It follows that $h_1(z)$ is continuous in the closure of $R\cdot\Pi_1$ on the boundary of $R\cdot\Pi_1$, and zero on the boundary of $R\cdot\Pi_1$, hence identically zero in $R\cdot\Pi_1$.

From the definitions and uniqueness of the functions involved we verify for t in $(R-R')\cdot\Pi_1$ and z in R the identity $g(z,t,R)-g(z,\bar{t},R)\equiv g(z,t,R\cdot\Pi_1)$, where the latter function is defined in $R\cdot\Pi_2$ by harmonic extension. Consequently equation (4) for z in $R \cdot \Pi_1$ can be extended so as to be valid for arbitrary z in R:

$$u(z) \equiv \int_{(R-R') \cdot \Pi_1} g(z, t, R)\, d\mu - \int_{(R-R') \cdot \Pi_2} g(z, t, R)\, d\mu,$$

where the integrals are taken over arbitrary disjoint open sets in R containing respectively $(R-R') \cdot \Pi_1$ and $(R-R') \cdot \Pi_2$. If as in (3) we now set $f(z) \equiv u(z) + iv(z)$, where $v(z)$ is conjugate to $u(z)$ in R', the conjugate of the vector $f'(z)$ represents the force at a point z of R' due to a negative distribution $-\mu$ on $(R-R') \cdot \Pi_1$ and an equal positive distribution on the inverse of $(R-R') \cdot \Pi_1$ with respect to C, plus the force at z due to the positive distribution μ on $(R-R') \cdot \Pi_2$ and an equal negative distribution on the inverse of $(R-R') \cdot \Pi_2$ in C. All these distributions are located on bounded point sets. Consequently the total force at 0 has a non-zero component vertically upward, as has the conjugate of $f'(z) = \partial u/\partial x + i \partial v/\partial x = \partial u/\partial x - i \partial u/\partial y$ at 0, and the theorem follows.

In Theorem 3, obviously no point of A in R can be a critical point of $u(z)$.

Theorem 3 can be used in proving

Theorem 4. *Let R be the interior of a Jordan curve C, and let B_1 and B_2 be two disjoint closed sets in R which together form the boundary of a subregion R' of R. Let $u(z)$ be harmonic in R', continuous in $R' + B_1 + B_2 + C$, equal to zero on C, to unity on B_1, and to $-c (< 0)$ on B_2. If a NE line λ for R separates B_1 and B_2, then no critical point of $u(z)$ lies on λ in R'. Consequently, if such lines λ exist, all critical points of $u(z)$ in R' lie in two NE convex closed subregions of R containing B_1 and B_2 respectively and separated by each NE line λ which separates B_1 and B_2.*

Map R onto the interior of the unit circle so that a given NE line λ is transformed into the axis of reals A and an arbitrarily chosen point of λ is transformed into the origin 0; we take B_1 in Π_1 and B_2 in Π_2, and we retain the original notation. It is sufficient to show that 0 is not a critical point. We define $u(z)$ as unity in the points of R separated from 0 by B_1, and as $-c$ in the points of R separated from 0 by B_2. Thus $u(z)$ is superharmonic in $R \cdot \Pi_1$ and subharmonic in $R \cdot \Pi_2$. Theorem 4 follows from Theorem 3.

Theorem 4 can be proved without the use of (4), but again the contrast between Theorems 3 and 4 indicates the power of the methods of RIESZ. Theorems 1 and 3 are presented here not to exhaust the range of the method but merely to illustrate the simplifying and unifying influence of the results due to F. RIESZ, by enabling us under broad conditions to set up and study a field of force corresponding to given superharmonic and subharmonic functions.

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