A remark on functions of several complex variables.

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1. Let $f(z)$ be a function of a complex variable z, regular for $|z|<1$. The very well known result of NEVALINNA and OSTROWSKI asserts that, if

$$\int_0^{2\pi} \log^+|f(re^{i\theta})|\,d\theta = O(1), \tag{1}$$

then $f(z)$ has a finite non-tangential limit at almost every point of the circumference $|z|=1$. Condition (1) is equivalent to the fact that the subharmonic and non-negative function $\log^+|f(z)|$ has a harmonic majorant for $|z|<1$. Since the existence of a harmonic majorant is an invariant of conformal mapping, the Nevanlinna—Ostrowski result may be stated for any domain limited by a simple and rectifiable curve, and even in a much more general case.

The situation is different for functions $f(z_1, z_2, \ldots, z_k)$ of several complex variables. Only in exceptional cases are two topologically equivalent domains in the space Z_k of the complex variables $z_1 = x_1 + iy_1, \ldots, z_k = x_k + iy_k$ equivalent through complex analytic mapping. Two simplest examples of such non-equivalent domains are the unit hypersphere

$$|z_1|^2 + |z_2|^2 + \ldots + |z_k|^2 < 1 \tag{S}$$

and the unit polycylinder

$$|z_1| < 1, |z_2| < 1, \ldots, |z_k| < 1. \tag{C}$$

As regards the latter, it has been shown (see [6]) that if $f(z_1, z_2, \ldots, z_k)$ is regular in C and if

$$\int_0^{2\pi} \ldots \int_0^{2\pi} \log^+|f|(\log^+\log^+|f|)^{k-1}\,d\theta_1 \ldots d\theta_k = O(1) \qquad (0 \leqq r_1, \ldots, r_k < 1), \tag{2}$$

where f stands for $f(r_1e^{i\theta_1}, r_2e^{i\theta_2}, \ldots, r_ke^{i\theta_k})$, then f has a finite limit for almost every $(\theta_1^0, \ldots, \theta_k^0)$ as the point $(r_1e^{i\theta_1}, \ldots, r_ke^{i\theta_k})$ approches $(e^{i\theta_1^0}, \ldots, e^{i\theta_k^0})$ along any non-tangential path. Though the problem is open, there seems to be little doubt that the factor $(\log^+\log^+|f|)^{k-1}$ in (2) cannot be omitted. This factor is analogous to the logarithmic factor in the theorem on the strong differentiability of multiple integrals and the latter is known to be indispensable (see [2], [5]).

The purpose of this brief note is to consider the case of the hypersphere S. It turns out that the iterated logarithm does not enter there and the situation resembles very much the Nevanlinna—Ostrowski theorem.

Theorem. *Let $f(z_1, z_2, \ldots, z_k)$ be regular in the hypersphere S and suppose that the integral*

$$\int_{\sigma_r} \log^+ |f(r_1 e^{i\theta_1}, \ldots, r_k e^{i\theta_k})| \, d\sigma_r \tag{3}$$

is bounded, where σ_r denotes the boundary of the hypersphere $r_1^2 + r_2^2 + \ldots + r_k^2 \leqq r^2$ and $d\sigma_r$ is the element of volume of this boundary. Then for almost every point $(z_1^0, z_2^0, \ldots, z_k^0)$ of σ_1 the function f has a finite non-tangential limit.

The proof is easy, if one uses a recent and important result of Calderón (see [1]) concerning boundary values of functions harmonic in a hypersphere. This result may be stated as follows. Suppose that $u(\xi_1, \xi_2, \ldots, \xi_n)$ is a real-valued harmonic function of the real variables $\xi_1, \xi_2, \ldots, \xi_n$ in the hypersphere

$$\xi_1^2 + \xi_2^2 + \ldots + \xi_n^2 < 1.$$

Suppose that for every point p_0 of a set E situated on the boundary of this hypersphere there is a (finite) cone with vertex at p_0, with axis along the radius terminating at p_0, and such that u is bounded in this cone. Then at almost every point of E the function u has a finite non-tangential limit. (For $n = 2$, this is an old result of Privaloff (see [3]), who proved it by the method of conformal mapping. For $n > 2$, however, the proof requires a totally different idea).

Let us now assume the boundedness of the integral (3) for $r < 1$. The function $\log^+ |f(x_1 + iy_1, \ldots, x_k + iy_k)|$ is in S a subharmonic function of the $2k$ variables $x_1, y_1, \ldots, x_k, y_k$. The proof of this is essentially the same as in the familiar case $k = 1$. First of all, $\log^+ |f|$ is continuous in S. It is therefore enough to show that for every point p_0 in S, and for every sufficiently small sphere S' with center at p_0, the value of $\log^+ |f|$ at p_0 does not exceed the average of $\log^+ |f|$ taken on the boundary of S'. If f vanishes at p_0, this is immediate, since $\log^+ |f|$ is non-negative. If f is distinct from 0 at p_0, then in a sufficiently small neighborhood of p_0 the function $\log |f|$ is harmonic in each pair of the variables x_i, y_j, and so also is harmonic in all the variables $x_1, y_1, \ldots, x_k, y_k$. Hence $\log^+ |f| = \mathrm{Max}\,\{0, \log |f|\}$ is subharmonic in that neighborhood, so that the required inequality is satisfied. Thus $\log^+ |f|$ is subharmonic in S.

Let $u_r(x_1, y_1, \ldots, x_k, y_k)$ be the Poisson integral formed with the values of $\log^+ |f|$ on σ_r. Thus u_r is a non-negative harmonic function in the interior of σ_r and majorizes $\log^+ |f|$ there. By a familiar result from the theory of subharmonic functions, $u_r(x_1, \ldots, y_k)$ is a non-decreasing function of r at every point $x_1, \ldots, y_k$ (of course for the values of r such that $r^2 > x_1^2 + \ldots + y_k^2$). By the theorem of Harnack, u_r tends in S either to a harmonic function u, or to $+\infty$. The latter is impossible since

$$\frac{1}{\text{meas } \sigma_r}\int_{\sigma_r} \log^+|f|\,d\sigma_r \tag{4}$$

represents the value of u_r at the origin and is a bounded function of r [the boundedness of (4) follows from the fact that it is a non-decreasing function of r and from the boundedness of the integral (3)].

Thus the function $\log^+|f(z_1,\dots,z_k)|$ has a harmonic majorant $u(z_1,\dots,z_k)$ in S. This harmonic function being non-negative, it is the Poisson integral of a positive mass distributed over the boundary σ_1 of S. Hence (as in the case $k=1$) $u(z_1,\dots,z_k)$ has a non-tangential limit at almost every point of σ_1. Since $u \geqq \log^+|f|$, it follows that f is at any rate bounded as $(z_1,\dots,z_k)$ approaches non-tangentially almost every point of σ_1. Since the real and the imaginary part of f are real-valued harmonic functions, an application of CALDERÓN's result shows that f has a non-tangential limit at almost every point of σ_1. This completes the proof the theorem.

2. Let $f(z_1,\dots,z_k)$ be a function regular in S, and let p be a positive number. As in the case $k=1$, we say that f belongs to the class H^p, if the integral

$$\int_{\sigma_r} |f|^p\,d\sigma_r \tag{5}$$

remains bounded for $r<1$. Since $\log^+|f| \leqq |f|^p + \text{Const.}$, the boundedness of the integral (5) implies that of (3). Thus if f is of the class H^p, the non-tangential limit of f exists at almost every point of σ_1, and is of course of the class L^p over σ_1. We shall denote this limit also by f. Let $F(z_1,\dots,z_n)$ denote the upper bound of $|f|$ on the radius of S terminating at the point $(z_1,\dots,z_k)\in\sigma_1$. It has recently been proved (see [4]) that if $f\in H^p$, then $F\in L^p$ on σ_1.

From this we immediately deduce the following

Theorem. *If* $f(z_1, z_2,\dots,z_k)\in H^p$, *then*

$$\int_{\sigma_1} |f(z_1,\dots,z_k)-f(rz_1,\dots,rz_k)|^p\,d\sigma_1 \to 0 \quad \textit{as } r\to 1.$$

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