Algebraic formulation of the problem of measure.

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Under a similar title, "Algebraische Fassung des Massproblems", ALFRED TARSKI has shown how the general problem of constructing an additive measure can be reduced to an interesting algebraic form[1]). In this paper we intend to carry TARSKI's reduction a little further[2]).

We consider, as TARSKI does, a non-void abstract set X with an arbitrary fixed triadic relation R, writing $R(x, y, z)$ to indicate that the elements x, y, z, of X are in the relation R. A non-negative real function μ defined on X will be called an R-additive measure if $R(x, y, z)$ implies $\mu(x) = \mu(y) + \mu(z)$. A suggestive notation consists in writing $x = y + z$ whenever $R(x, y, z)$; but the operation $+$ thus introduced need not be defined for all pairs (y, z), need not be single-valued, and need not enjoy any special algebraic properties. Illustrations and interpretations of these concepts may be found in TARSKI's paper[1]). A particularly interesting example arises in abstract geometry: let the geometrical structure of a space be defined in terms of a relation of congruence among its subsets, let X be a family of subsets of the given space, and let $R(x, y, z)$ if and only if x is the union of disjoint sets congruent respectively to y and to z; then an R-additive measure defined over X will in general have the characteristic properties of a geometrical content, being additive for the union of disjoint sets and invariant under the replacement of a set by a congruent one.

Since the real number system can be viewed as a rational-linear space, that is, as a linear space with the rational number system as coefficient-field, we may regard an R-additive measure as merely a special instance of an R-additive mapping of X into a rational-linear space. Our first step will be to analyze the structure of such general mappings. It is comparatively easy to show that any such mapping decomposes into a fixed mapping of X into

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[1]) A. TARSKI, *Fundamenta Math.*, **31** (1938), pp. 47–66.

[2]) Our results were included in Colloquium Lectures delivered before the American Mathematical Society at Madison, Wisconsin, in September, 1939. They have also been presented in lectures at Harvard University (1945–46) and the University of Chicago (1946).

a certain fixed rational-linear space L_X and a rational-linear mapping of the space L_X. The problem of constructing an R-additive measure is thus reduced to the problem of constructing a rational-linear mapping of L_X into the real number system, subject to certain requirements of positivity. The latter problem can then be solved in terms of the theory of convex sets in a rational-linear space.

The rational-linear space L_X and the natural R-additive mapping T of X into L_X will now be constructed. Let L be the rational-linear space of all rational-valued functions f on X such that the set $(x; f(x) \neq 0)$ is finite. Let f_x designate that member of L defined by the relations $f_x(x) = 1$, $f_x(y) = 0$ when $y \neq x$. The mapping H: $x \to f_x$ carries X in a one-to-one manner onto a set of rationally-linearly independent elements in L. Let L_0 be the rational-linear subspace of L generated by the functions of the special form $f_x - f_y - f_z$ where $R(x, y, z)$; an element of L belongs to L_0 if and only if it is a rational-linear combination of a finite number of these special functions. Identification of the elements of L modulo L_0 in the standard way produces a rational-linear space $L_X = L - L_0$, and can be regarded as a rational-linear mapping of L on L_X. We shall denote this mapping as G and refer to it as the natural mapping of L on L_X. The mapping $T = GH$ is a mapping of X into L_X which is R-additive in the sense that $R(x, y, z)$ implies $Tx = Ty + Tz$: for $R(x, y, z)$ implies that the elements f_x and $f_y + f_z$ are to be identified modulo L_0 in as much as their difference is in L_0; and thus $Tx = G(Hx) = Gf_x = G(f_y + f_z) = Gf_y + Gf_z = G(Hy) + G(Hz) = Ty + Tz$.

We now have:

T h e o r e m 1. *If M is a rational-linear space and A an R-additive mapping of X into M (in the sense that $R(x, y, z)$ implies $Ax = Ay + Az$), then there exists a rational-linear mapping S of L_X into M such that $A = ST$. Conversely, if S is any rational-linear mapping of L_X into M, then the mapping $A = ST$ of X into M is R additive.*

P r o o f. Consider the mapping AH^{-1}, which carries f_x into Ax. Since every element of L is a rational-linear combination of the rationally-linearly independent elements f_x, this mapping has a unique rational-linear extension U which maps L into M: U carries the element $\sum_{k=1}^{n} \alpha_k f_{x_k}$ of L into the element $\sum_{k=1}^{n} \alpha_k Ax_k$. Now $R(x, y, z)$ implies $U(f_x - f_y - f_z) = Uf_x - Uf_y - Uf_z = Ax - Ay - Az = 0$. It follows that U carries every element of the rational-linear subspace L_0 into the element 0 of M. Consequently U can be decomposed as a product $U = SG$ where S is a rational-linear mapping of $L_X = L - L_0$ into M and G is the natural mapping of L on L_X. We now have $A = UH = SGH = ST$, as we wished to prove. The converse statement is trivial.

Corollary. *If M is the real number system, considered as a rational-linear space, then the mapping A of Theorem 1 is an R-additive measure if and only if the mapping S is a rational-linear real-valued functional which assumes only non-negative values on the subset $T(X)$ of L_X.*

From Theorem 1 and its corollary, we see that the problem of finding the R-additive measures for X has been reduced to an algebraic problem about rational-linear spaces. We shall now discuss the latter problem in some detail. Let V be a rational-linear space with elements $u, v, w, \ldots$; and let V_0 be an arbitrary non-void subset of V. Our problem, to state it in one way, is to construct a real-valued function λ on V with the properties

(1) $\lambda(u+v)=\lambda(u)+\lambda(v)$, (2) $\lambda(u) \geqq 0$ for $u \in V_0$.

To eliminate trivial solutions we shall also require that

(3) $\lambda(v_0)>0$ for some $v_0 \in V_0$.

From (1) we infer at once, in a familiar way, that λ is rational-linear, having the property that $\lambda(\alpha u)=\alpha\lambda(u)$ for all rational α. Let us consider what implications the existence of such a function λ may have for the relative positions of v_0 and V_0 in the rational-linear geometry of V. For this purpose we shall need some simple definitions. First, let us define a subset of V to be *convex* if, whenever it contains u and v, it also contains $\alpha u+\beta v$ where α and β are any *rational* numbers such that $\alpha \geqq 0$, $\beta \geqq 0$, $\alpha+\beta=1$. Similarly, let us define a subset of V to be a *cone with vertex* u_0 if, whenever it contains u, it also contains $\alpha u+\beta u_0$ where α and β are any *rational* numbers such that $\alpha \geqq 0$, $\alpha+\beta=1$. A convex cone with vertex at the origin is thus characterized by the property that, wheneve it contains u and v, it also contains $\alpha u+\beta v$ where α and β are any non-negative *rational* numbers. *In the sequel we shall use the term "cone" to mean always a convex cone with vertex at the origin,* no other type of cone being required for our purposes. Any non-void subset of V is contained in a smallest cone, consisting of all the linear combinations with non-negative rational coefficients of the element of the given subset. Finally we shall define a point u_0 beloging to a set U to be *internal* to U if for each $u \neq u_0$ the element $\alpha u+\beta u_0$, where α and β are rational numbers with $\alpha \geqq 0$, $\beta \geqq 0$, $\alpha+\beta=1$, belongs to U not merely for $\alpha=0$ but also for all sufficiently small $\alpha>0$. Geometrically this means that every straight line through u_0 has in common with U a certain segment on which u_0 lies but of which it is not an end-point. Consider now the smallest cone $C(V_0)$ containing V_0, and the set $K=(u\,;\lambda(u)<0)$. It is clear that λ is non-negative on $C(V_0)$ as well as on V_0, and hence that K is disjoint from $C(V_0)$. At the same time K is obviously a convex set containing $-v_0$. Indeed, it is easy to see that $-v_0$ is internal to K: for if $u \neq -v_0$ we have $\lambda(\alpha u\ \beta v_0)=\alpha\lambda(u)-\beta\lambda(v_0)<0$ for all sufficiently small rational $\alpha>0$ when $\alpha+\beta=1$. These necessary properties can now be shown to be sufficient as well:

Theorem 2. *A necessary and sufficient condition for the existence of a function λ with properties* (1), (2), (3) *above is that $-v_0$ be internal to some convex set K disjoint from the smallest cone containing V_0.*

Proof. Only the sufficiency remains to be proved. ZORN's maximal principle[3]), is applicable to the system of cones containing V_0 and disjoint from K. Thus there is a maximal such cone, say S_0. An important property of S_0 is that it must contain at least one of the two elements u and $-u$, $u \in V$. To prove this, let us observe that the smallest cone containing the element u and the maximal cone S_0 consists of all elements of the form $\alpha u + v$ where α is a non-negative rational number and $v \in S_0$. Hence if u is not in S_0 this cone must have a point in common with K because of the maximality of S_0; in other words, there exist a positive rational α and an element v in S_0 such that $\alpha u + v \in K$. Similarly, if $-u$ is not in S_0 there exist a positive rational β and an element w in S_0 such that $-\beta u + w \in K$. If neither u nor $-u$ is in S_0 we obtain a contradiction as follows: putting $\alpha' = \beta/(\alpha+\beta)$, $\beta' = \alpha/(\alpha+\beta)$, we see that these positive rational numbers have 1 as sum; the convexity of K thus implies $\alpha' v + \beta' w = \alpha'(\alpha u + v) + \beta'(-\beta u + w) \in K$; but on the other hand, the convexity of S_0 implies $\alpha' v + \beta' w \in S_0$. In particular we see that the set $-K = (u;\ -u \in K)$ is contained in S_0 and obviously contains v_0 as an internal point. Hence S_0 also contains v_0 as an internal point. A second application of ZORN's principle provides us now with a maximal cone S containing S_0 and excluding $-v_0$. Clearly S, like its subset S_0, contains at least one of u and $-u$, $u \in V$. Since $U = (u;\ u \in S,\ -u \in S)$ is obviously a rational-linear subspace of V, we may effect the identification of elements in V modulo U so as to obtain the rational-linear space $W = V - U$. The natural mapping of V on W will be designated as λ. The space W can be ordered by putting $\lambda(u) < \lambda(v)$ if and only if $v - u \in S$, $u - v$ non $\in S$. In this ordering we have $\lambda(v_0) > 0$, $\lambda(u) \geqq 0$ for $u \in S$. Thus, if we can show that the order in W is archimedean, we can identify W with a rational linear subspace of the real number system, ordered in the standard way; and we can identify λ as the function which we desired to construct. Hence all that remains for us to do is to show that $\lambda(u) > 0$ implies the existence of positive rational numbers α and β for which $\alpha\lambda(v_0) \leqq \lambda(u) \leqq \beta\lambda(v_0)$. To find α, we observe that $-u$ non $\in S$ and hence that there exist a positive rational number γ and an element v in S for which $-\gamma u + v = -v_0$, in accordance with the maximality of the cone S. Taking $\alpha = 1/\gamma$, we have $u - \alpha v_0 = \dfrac{1}{\gamma} v \in S$ and hence $\lambda(u) \geqq \lambda(\alpha v_0) = \alpha\lambda(v_0)$. To find β, we note that for sufficiently small positive

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[3]) This principle asserts that any non-void partially ordered system in which every chain is bounded has at least one maximal element. In our case we consider the system of cones to be partially ordered by inclusion.

rational α we have $\alpha(-u)+(1-\alpha)v_0 \in S$ because v_0 is internal to S. Then taking $\beta=(1-\alpha)/\alpha$ we have $\beta v_0 - u \in S$ and hence $\beta\lambda(v_0) \geqq \lambda(u)$. This completes the proof [4]).

Inspection of the proof just given shows that we can state

Theorem 3. *For the existence of a function λ with the properties* (1), (2), (3) *above it is sufficient that the smallest cone containing V_0 contain v_0 as an internal point but exclude $-v_0$.*

Using Theorem 3 it is easy to see that under certain circumstances the problem of the existence of an R-additive measure for X becomes a purely combinatorial one. We have

Theorem 4. *Let x_0 be such an element of X that Tx_0 is an internal point of the smallest cone containing TX in L_X. Then a necessary and sufficient condition for the existence of an R-additive measure μ for X with $\mu(x_0)>0$ is that for every finite subset Y of X containing x_0 there exist an R-additive measure μ_Y with $\mu_Y(x_0)>0$.*

Proof. The necessity of the condition is trivial since we can put $\mu_Y(x) = \mu(x)$ for all $x \in Y$ when the R-additive measure μ is known to exist. To prove the condition sufficient, we derive from it the result that $-Tx_0$ does not belong to the smallest cone containing TX; and we can then apply Theorem 3. If $-Tx_0$ belongs to the indicated cone, we may observe that this cone is the image under the natural mapping G of the smallest cone containing HX in L and hence that the latter cone must contain an element f which is to be identified modulo L_0 with $Hx_0 = -f_{x_0}$. We note that the function f is non-negative and that there exist rational numbers α_k and elements x_k, y_k, z_k in X with $R(x_k, y_k, z_k)$ such that $f_{x_0}+f=g=\sum_{k=1}^{n}\alpha_k(f_{x_k}-f_{y_k}-f_{z_k}) \in L_0$. Let Y be the set consisting of $x_0, x_1, y_1, z_1, \ldots, x_n, y_n, z_n$ and those elements x, finite in number, for which $f(x) \neq 0$. The rational-linear space of rational-valued functions defined on Y is isomorphic to the rational-linear subspace of L generated by $HY \subset HX$; and the rational-linear space L_Y associated with Y can be identified with the rational-linear subspace of L_X generated by $TY \subset TX$. The existence of the R-additive measure μ_Y with $\mu_Y(x_0)>0$ implies, by Theorem 1, the existence of a rational-linear real function λ_Y on L_Y such that λ_Y is non-negative on TY and $\lambda_Y(Tx_0)>0$. Observing now that $Gf_{x_0}=Tx_0$, Gf is in the smallest cone containing TY, and $Gf_{x_0}+Gf = Gg = 0$, we arrive at the absurd result $0<\lambda_Y(Tx_0)+\lambda_Y(Gf)=\lambda_Y(0)=0$.

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[4]) In this presentation, we have taken advantage of a short paper of J. Dieudonné, Sur le théorème de Hahn—Banach, *Revue Scientifique,* **79** (1941), pp. 642—643, where the author treats the Hahn—Banach theorem in terms of the theory of convexity in a real linear space.

Hence we see that $-Tx_0$ is outside the smallest cone containing TX, as we asserted above. This completes the proof.

By way of conclusion we may make two remarks. In the first place, G. MOSTOW[5]) has observed that in terms of a certain "natural" topology for rational-linear spaces Theorem 2 may be given the equivalent form: the required function λ exists if and only if $-v_0$ is not a point of the smallest closed cone containing V_0. The topology of MOSTOW is that in which a set is said to be open if and only if each of its points is internal to some convex part of the set. We shall not pursue this remark further here. Our second remark is the rather obvious one that the general theory developed in TARSKI's paper and in this one needs to be tested on specific examples. The discussion in TARSKI's paper shows that the construction of the space L_X conceals apparently difficult combinatorial problems met in determining whether or not certain elements in L are to be indentified modulo L_0; and this fact suggests the difficulties which can be anticipated in trying to ascertain the relative positions of Tx_0 and TX in L_X in any concrete case. Furthermore, the observation that in some simple cases, familiar to everyone, the R-additive measure for X is essentially unique leads to an inquiry as to the conditions on TX which will guarantee uniqueness in general terms.

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[5]) Mr. MOSTOW was a member of a Harvard class to which I presented Theorem 2 in 1945–46. I have since made systematic use of his topology in developing the theory convexity relative to an arbitrary ordered field (unpublished).