Uniform Distribution and Lebesgue Integration.

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1. If $u_1, u_2, \ldots$ denotes a sequence of real numbers uniformly distributed modulo 1 and if $f(x)$ is a bounded Riemann-integrable function of the real variable x, with period 1, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(u_n) = \int_0^1 f(t)\,dt.$$

It is obvious that the theorem becomes false if, instead of supposing that f is Riemann-integrable, we assume only that f is Lebesgue-integrable, since we can change arbitrarily the values of f at all points u_n (mod 1) without changing the integral.

A natural question to ask is whether for $f \in L$, the relation

$$\text{(1)} \qquad \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x+u_n) = \int_0^1 f(t)\,dt$$

holds almost everywhere in x. If $u_n = \theta n$, where θ is any fixed irrational number, the relation (1) holds for almost all x, under the only assumption that $f \in L$. This result, due to KHINTCHINE[1]) is actually an instance of BIRKHOFF's ergodic theorem[2]), and one cannot expect a generalization of the argument to general uniformly distributed sequences.

Here, using an argument based on different ideas, we shall give some results of the type (1), confining ourselves to the case $f \in L^2$ and to certain types of sequences $\{u_n\}$.

If, instead of a result of the type (1) we consider convergence in mean, we can state the following general theorem[3]):

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[1]) A. KHINTCHINE, Eine arithmetische Eigenschaft der summierbaren Funktionen, *Recueil Math. Moscou*, 41 (1934), pp. 11—13.

[2]) For litterature see [1]).

[3]) This theorem, the proof of which is very simple, may be known but we did not find it in the litterature.

Theorem I. *Let $f(x) \in L^2$ be a function with period 1 and mean value zero, i. e. $\int_0^1 f(x)\,dx = 0$. Then, for any sequence $\{u_n\}$ uniformly distributed modulo 1, one has*

$$\lim_{N\to\infty} \int_0^1 \frac{1}{N} \left| \sum_{n=1}^{N} f(x+u_n) \right|^2 dx = 0.$$

Proof. Let $\sum_{-\infty}^{\infty} c_k e^{2\pi i k x}$ be the Fourier series of $f(x)$, $(c_0 = 0,\ c_{-k} = \bar{c}_k)$. Let us write

$$S_k = \frac{1}{N}(e^{2\pi i k u_1} + \ldots + e^{2\pi i k u_N}),$$

so that the integral considered in the theorem is equal to

$$2 \sum_{1}^{\infty} |c_k|^2 |S_k|^2$$

and, since $|S_k| \leqq 1$, does not exceed

$$2 \sum_{1}^{h} |c_k|^2 |S_k|^2 + 2 \sum_{h+1}^{\infty} |c_k|^2.$$

If we now choose h such that $\sum_{h+1}^{\infty} |c_k|^2 < \varepsilon$ $(\varepsilon > 0)$, and then N_0 such that $|S_k|^2 < \varepsilon$ for $k = 1, 2, \ldots, h$ and $N \geqq N_0$ the integral will not exceed

$$\varepsilon \left[\int_0^1 f^2\,dx + 2 \right]$$

for $N \geqq N_0$, which proves the theorem.

2. We are unable to state a result of the type (1) without making certain additional hypotheses on the function f and on the sequence $\{u_n\}$. (That some additional hypotheses, at least on the function f, are necessary, will be shown at the end of the paper, with the use of an argument due to Erdős).

Let again $f \in L^2$ have period 1 and mean value zero, so that

$$f(x) \sim \sum_{-\infty}^{\infty} c_k e^{2\pi i k x} \qquad (c_0 = 0,\ c_{-k} = \bar{c}_k).$$

Let us denote by $R(h)$ the remainder $\sum_{h+1}^{\infty} |c_k|^2$.

Let us now denote by $S(M, N, k)$ the sum

$$\sum_{n=M+1}^{M+N} e^{2\pi i k u_n} \qquad (M, N \text{ and } k \text{ being integers}).$$

We can state the following theorem:

Theorem II. *Let $f \in L^2$ have period 1 and mean value zero, and be such that $R(h) = O\left(\frac{1}{(\log h)^\alpha}\right)$ where $\alpha > 1$. Let $\{u_n\}$ be a sequence uniformly distributed modulo 1 such that*

$$|S(M, N, k)| \leqq \varDelta k^\varrho N^\sigma (M+N)^\tau \qquad (k \geqq 1,\ M \geqq 1,\ N \geqq 1),$$

where $\varDelta$, ϱ, σ, τ are constants such that $\sigma + \tau < 1$ and $\tau < 1/2$. Then, almost everywhere in x,

$$\lim_{N \to \infty} \frac{1}{N} [f(x+u_1) + \ldots + f(x+u_N)] = 0.$$

Remark. As $f(x)$ needs not be bounded, theorem II is applicable to certain periodical functions which are only improperly integrable in the sense of RIEMANN.

The proof depends on the following lemma, which is a particular case of a result of GÁL and KOKSMA[4]). We give here a proof somewhat different from the original one.

Lemma: *Let $\{f_\nu(x)\}$, $\nu = 1, 2, \ldots$ be a sequence of functions all belonging to L^p $(p > 1)$ in the interval $(0, 1)$. Let $\eta(N)$ be positive monotonic decreasing such that $\sum \frac{\eta(N)}{N} < \infty$. Suppose that for all $M \geqq 0$, $N \geqq 1$*

$$\int_0^1 \left| \sum_{\nu=M+1}^{M+N} f_\nu \right|^p dx \leqq C (M+N)^{p-\lambda} N^\lambda \eta(N)$$

where $\lambda > 1$. Then, for almost all x,

$$\lim_{N \to \infty} \frac{1}{N} (f_1 + f_2 + \ldots + f_N) = 0.$$

Proof of the lemma. Let n be a positive integer. By $\varDelta_k^{(h)}$ $(h = 1, 2, \ldots, 2^k)$ we denote any of the intervals (open on the left, closed on the right) obtained by the subdivision of the interval $(0, 2^n)$ in 2^k equal parts. By $S_k^{(h)}$ we denote the sum Σf_ν where ν takes all integral values contained in $\varDelta_k^{(h)}$.

Denoting by j any fixed integer such that $1 \leqq j \leqq 2^n$, and writing j in the dyadic system, we find that the interval $(0, j)$ is the sum of certain intervals $\varDelta_k^{(h)}$ where k takes at most once each value $0, 1, 2, \ldots, n$, and each h depends on the corresponding k. According to this

$$\sum_{\nu=1}^{j} f_\nu = \varepsilon_0 S_0^{(h_0)} + \ldots + \varepsilon_n S_n^{(h_n)},$$

where $\varepsilon_i = 0$ or 1.

[4]) I. S. GÁL and J. F. KOKSMA, Sur l'ordre de grandeur des fonctions sommables, *Comptes Rendus Acad. Sci. Paris*, **227** (1949), pp. 1321—1323. The complete proof of the general theorem will appear elsewhere.

Let θ be a positive number larger than 1, to be fixed later on, one has, using HÖLDER's inequality,

$$\left|\sum_{\nu=1}^{j} f_\nu\right|^p \leqq \left(\sum_{k=0}^{n} \frac{1}{\theta^k}\,\theta^k\,|S_k^{(h_k)}|\right)^p \leqq \left(\sum_{k=0}^{n} \frac{1}{\theta^{p'k}}\right)^{p-1} \left(\sum_{k=0}^{n} \theta^{pk}|S_k^{(h_k)}|^p\right),$$

where $\frac{1}{p}+\frac{1}{p'}=1$. Hence, for *all* j $(1 \leqq j \leqq 2^n)$ and *all* x

$$\left|\sum_{\nu=1}^{j} f_\nu\right|^p \leqq B \sum_h \sum_k \theta^{pk}|S_k^{(h)}|^p,$$

where $B=\left(\sum_{k=0}^{\infty} \frac{1}{\theta^{p'k}}\right)^{p-1}$, and the double summation is extended to $k=0, 1, 2, \ldots, n$, and for each k to *all* values of h $(h=1, 2, \ldots, 2^k)$. Now, by hypothesis,

$$\int_0^1 |S_k^{(h)}|^p\,dx \leqq C 2^{n(p-\lambda)} 2^{(n-k)\lambda} \eta(2^{n-k}).$$

Hence

$$\int_0^1 \left|\sum_{\nu=1}^{j(x)} f_\nu\right|^p dx \leqq B \sum_{k=0}^{n} \theta^{pk}.\, 2^k C 2^{n(p-\lambda)} 2^{(n-k)\lambda} \eta(2^{n-k}),$$

where we can suppose that the integer $j(x)$ is any measurable function of x. Supposing now $2^{n-1} < j(x) \leqq 2^n$, one has

$$\int_0^1 \left|\frac{\sum_{\nu=1}^{j(x)} f_\nu}{j(x)}\right|^p dx = O\left\{\frac{1}{2^{pn}} \sum_{k=0}^{n} \theta^{pk} 2^k 2^{n(p-\lambda)} 2^{(n-k)\lambda} \eta(2^{n-k})\right\}$$

$$= O\left\{\sum_{k=0}^{n} \frac{\theta^{pk}}{2^{(\lambda-1)k}} \eta(2^{n-k})\right\}.$$

Now fix θ such that $1 < \theta < 2^{\frac{\lambda-1}{p}}$ (which is possible since $\lambda > 1$), and put

$$\alpha = \frac{\theta^p}{2^{\lambda-1}} < 1\,;$$

one has

$$\sum_{k=0}^{n} \alpha^k \eta(2^{n-k}) = \sum_{0}^{\left[\frac{n}{2}\right]} + \sum_{\left[\frac{n}{2}\right]}^{n} = O(\eta(2^{n/2})) + O(\alpha^{n/2}),$$

and, remarking that the condition $\sum \frac{\eta(N)}{N} < \infty$ implies $\sum \eta(2^{n/2}) < \infty$, one has, writing

$$I_n = \int_0^1 \left|\frac{\sum_{\nu=1}^{j(x)} f_\nu}{j(x)}\right|^p dx \qquad (2^{n-1} < j(x) \leqq 2^n),$$

that $\Sigma I_n < \infty$. In other words,

$$\sum_{n=1}^{\infty} \int_0^1 \max_{2^{n-1}<j\leqq 2^n} \left| \frac{\sum_{\nu=1}^{j} f_\nu}{j} \right|^p dx < \infty,$$

which implies

$$\sum_1^N f_\nu = o(N)$$

for almost all x.

P r o o f o f T h e o r e m II. Writing

$$T_{M,N} = \int_0^1 \left| \sum_{n=M+1}^{M+N} f(x+u_n) \right|^2 dx$$

one has, using the hypotheses of the theorem:

$$\begin{aligned} T_{M,N} &= 2\sum_{k=1}^{\infty} |c_k|^2 |S(M,N,k)|^2 \leqq \\ &\leqq 2A^2 \sum_{k=1}^{h} |c_k|^2 k^{2\varrho} N^{2\sigma} (M+N)^{2\tau} + 2N^2 \sum_{k=h+1}^{\infty} |c_k|^2 \leqq \\ &\leqq A' \left[h^{2\varrho} N^{2\sigma} (M+N)^{2\tau} + \frac{N^2}{(\log h)^\alpha} \right], \end{aligned}$$

A' being a constant. Fix now an ε, positive, such that

$$2\varrho\varepsilon + 2\sigma + 2\tau < 2 \tag{2}$$

as is clearly possible since $\sigma + \tau < 1$, and take for h the integral part of N^ε. Then

$$T_{M,N} \leqq C\left[N^{2\varrho\varepsilon+2\sigma} (M+N)^{2\tau} + \frac{N^2}{(\log N)^\alpha} \right],$$

C being a constant. Writing

$$T_{M,N} \leqq C\left[\frac{(M+N)^{2\tau} N^{2-2\tau}}{N^{2-2\tau-2\varrho\varepsilon-2\sigma}} + \frac{N^2}{(\log N)^\alpha} \right],$$

one has by (2)

$$T_{M,N} \leqq D \frac{(M+N)^{2\tau} N^{2-2\tau}}{(\log N)^\alpha},$$

D being a constant. Since $\tau < 1/2$, $\alpha > 1$, an application of the lemma (with $p = 2$) gives

$$\lim_{N\to\infty} \frac{1}{N} [f(x+u_1) + f(x+u_2) + \ldots + f(x+u_N)] = 0$$

for almost all x.

3. Applications. We propose now to give examples of sequences $\{u_n\}$ uniformly distributed (mod 1) for which the relation

$$|S(M, N, k)| \leqq A k^{\varrho} N^{\sigma} (M+N)^{\tau} \quad (\sigma+\tau<1, \ \tau<1/2)$$

is satisfied.

First Example. Let θ denote an irrational number of the type I, that is to say that for some constant $\eta>2$, the inequality

$$\left|\theta - \frac{p}{q}\right| < \frac{1}{q^{\eta}}$$

has only a finite number of solutions in integers p and q, $(q>1)$. We can take, for instance, for θ any algebraic number; or any irrational number with bounded partial quotients. By a well known theorem the numbers which are not of the type I form a null set (BOREL).

Let now, r being an integer $\geqq 2$,

$$u_n = \theta n^r + \alpha_1 n^{r-1} + \ldots + \alpha_r,$$

where $\alpha_1, \ldots, \alpha_r$ are arbitrary real constants. We shall prove that for the sequence $\{u_n\}$ using the notations of theorem II, one has

$$(3) \qquad |S(M, N, k)| \leqq A k^{\varrho} N^{\sigma} \qquad (\sigma<1)$$

so that theorem II is applicable to such a sequence.

In fact, this can be deduced from theorems of WEYL, VINOGRADOFF and others. As we do not need the modern results in their sharpest form, we make use, instead, of the following special case of a theorem of KOKSMA[5]), which has the advantage that the wanted inequality (3) follows from it immediately:

Let r denote a positive integer; put $P=2^r$; θ is an irrational number of the type I, described above, so that a number $L=L(\theta)$ exists such that for all integers $q>1$,

$$|\sin \pi q \theta| > \frac{L}{q^{\eta-1}}.$$

Then if $\varphi(n)$ denotes the polynomial $k u_n$, we have[6])

$$\frac{1}{N}\left|\sum_{M+1}^{M+N} e^{2\pi i \varphi(\nu)}\right| \leqq 50 \left(\frac{k^{\eta-1}(r!)^{\eta-1}}{LN}\right)^{\frac{1}{(P-2)(\eta-1)+\frac{P}{2}}}.$$

From this, (3) follows with $\varrho=\eta-1$ and $\sigma<1$.

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[5]) J. F. KOKSMA, Over stelsels Diophantische Ongelijkheden, *Dissertation Groningen*, 1930, Theorem (Stelling) 10, p. 61.

[6]) For the convenience of the reader, this result is obtained by taking the one-dimensional case in KOKSMA's theorem (see [5])) with

$$\Theta=\theta,\ f=\varphi=k u_n,\ g=r!\,k\theta,\ t=1,\ d=\eta-1,\ h=kr!$$

and $R=\Delta^r f - g = 0$.

Second example. Let $f(t)$ be a p-times differentiable function $(p \geqq 2)$ for $t \geqq 1$, such that $f^{(p)}(t)$ has the same sign for all t, and that

$$\frac{c}{t^{1-\gamma}} \leqq |f^{(p)}(t)| \leqq \frac{C}{t^{1-\gamma}} \qquad (0<\gamma<1,\ 0<c<C),$$

where c, C and γ are independent of t. Then for the sequence $u_n = f(n)$ one has

$$|S(M, N, k)| \leqq A k^{\varrho} N^{\sigma} (M+N)^{\tau} \tag{4}$$

with $\sigma+\tau<1$, $\tau<1/2$, so that theorem II is applicable to the sequence $\{u_n\}$.

The proof of (4) is based on the following lemma of VAN DER CORPUT[7]):

Lemma. *Let $M \geqq 0$ $N \geqq 1$, $p \geqq 2$ be all integers, put $P=2^p$ and let $g(t)$ be a real function for $M \leqq t \leqq M+N$ which admits a derivative of order p, say $g^{(p)}(t)$ and suppose that $g^{(p)}(t) \geqq r$ for all t, or $g^{(p)}(t) \leqq -r$ for all t, where r is independent of t. Writing*

$$R = \frac{1}{N} |g^{(p-1)}(M+N) - g^{(p-1)}(M)|$$

one has

$$\left|\sum_{n=M}^{M+N} e^{2\pi i g(n)}\right| \leqq 21 N \left\{ \left(\frac{r}{R^2}\right)^{-\frac{1}{P-2}} + (rN^p)^{-\frac{2}{P}} + \left(\frac{rN}{R}\right)^{-\frac{2}{P}} \right\}. \tag{5}$$

Now apply the lemma to the function $g(t) = kf(t)$, where $f(t)$ satisfies the conditions of our example, and put

$$r = \frac{ck}{(M+N)^{1-\gamma}}, \quad R = \frac{1}{N} \left| \int_M^{M+N} k f^{(p)}(t)\, dt \right|$$

so that

$$R \leqq \frac{1}{N} \int_M^{M+N} \frac{Ck\,dt}{t^{1-\gamma}} \leqq \frac{Ck}{N} \int_0^N t^{\gamma-1} dt = \frac{Ck}{\gamma} \frac{1}{N^{1-\gamma}}.$$

We have now, c_1, c_2, etc. being constants:

$$\left(\frac{r}{R^2}\right)^{-\frac{1}{P-2}} \leqq c_1 k^{\frac{1}{P-2}} (M+N)^{\frac{1-\gamma}{P-2}} N^{-\frac{2(1-\gamma)}{P-2}},$$

$$(rN^p)^{-\frac{2}{P}} \leqq c_2 k^{-\frac{2}{P}} (M+N)^{\frac{2(1-\gamma)}{P}} N^{-\frac{2p}{P}},$$

$$\left(\frac{r}{R} N\right)^{-\frac{2}{P}} \leqq c_3 (M+N)^{\frac{2(1-\gamma)}{P}} N^{-\frac{2}{P}(2-\gamma)}.$$

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[7]) See e. g. J. G. VAN DER CORPUT, Neue zahlentheoretische Abschätzungen. II, *Math. Zeitschrift*, **29** (1929), pp. 397–426.

Hence, by (5)

$$\left|\sum_{n=M}^{M+N} e^{2\pi i k f(n)}\right| \leqq c_4 k^{\frac{1}{P-2}} (M+N)^{\frac{2(1-\gamma)}{P}} N^{1-\frac{2(1-\gamma)}{P-2}},$$

the inequality being obtained by remarking that, since $0 < \gamma < 1$, $p \geqq 2$, $P \geqq 4$, one has

$$\frac{2(1-\gamma)}{P} \geqq \frac{1-\gamma}{P-2}$$

and

$$\frac{2(1-\gamma)}{P-2} < \frac{2(2-\gamma)}{P} < \frac{2p}{P}.$$

Writing now $\varrho = \frac{1}{P-2}$, $\sigma = 1 - \frac{2(1-\gamma)}{P-2}$, $\tau = \frac{2(1-\gamma)}{P}$, we remark that, since $P \geqq 4$, $0 < \gamma < 1$, one has $\tau < 1/2$ and

$$\sigma + \tau = 1 - \frac{2(1-\gamma)}{P-2} + \frac{2(1-\gamma)}{P} < 1$$

so that

$$\left|\sum_{n=M}^{M+N} e^{2\pi i k f(n)}\right| \leqq c_4\, k^{\varrho} N^{\sigma} (M+N)^{\tau}$$

with $\sigma + \tau < 1$, $\tau < 1/2$. We conclude that, under the conditions stated for $f(t)$, Theorem II is applicable to the sequence $u_n = f(n)$.

4. In view of Theorem II the question arises, whether by imposing to the sequence $u_1, u_2, \ldots$ sufficiently strong conditions, e. g. with respect to its discrepancy[8]) $D(N)$, one could avoid any sort of condition on the Fourier coefficients of $f(x)$ and have the relation (1) by merely supposing that the periodic function f belongs to $L^{(2)}$. The answer to this question is negative, as follows from an interesting counterexample due to P. Erdős who communicated it to us verbally: *For every given positive number* $\varepsilon < 1$ *and every decreasing sequence of positive numbers* $\{\delta_n\}$ *for which*

$$\sum_{n=1}^{\infty} \delta_n < \varepsilon \tag{6}$$

a function $f(x)$ *on* $(0, 1)$ *can be constructed, which takes the values* 0 *and* 1 *only, for which* $\int_0^1 f(x)\,dx < \varepsilon$, *whereas the following assertion holds: If* $u_1, u_2, \ldots$ *is any sequence on* $(0, 1)$, *then it can be replaced by a sequence* $u'_1, u'_2, \ldots$ *such that*

$$|u'_n - u_n| < \delta_n \; (n \geqq 1)$$

[8]) For the definition of discrepancy see e. g. J. F. Koksma, Diophantische Approximationen, *Ergebnisse der Math. und ihrer Grenzgebiete,* IV. 4 (Berlin, 1936), Kap. VIII § 2, p. 90.

whereas for all x

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(u'_n + x) = 1.$$

Now it is obvious that, if the sequence $u_1, u_2, \ldots$ is uniformly distributed (mod 1) with the discrepancy $D(N)$, we can choose $\delta_1, \delta_2, \ldots$ so rapidly decreasing that the sequence $u'_1, u'_2, \ldots$ is also uniformly distributed and has the discrepancy $\leqq 4D(N)$. Therefore:

No matter how fast the positive decreasing function $\varphi(N)$ may turn to zero as $N\to\infty$, if there are sequences $u_1\, u_2, \ldots$ for which $D(N)\leqq\varphi(N)$, there exist a function $f(x)\in L^2$ and certain sequences $u'_1, u'_2, \ldots$ satisfying $D(N)\leqq 4\varphi(N)$, such that we have

$$\int_0^1 f(x)\,dx < 1/2 \quad \textit{and} \quad \limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(u'_n + x) = 1$$

for every x on $(0, 1)$.

We give a complete sketch of the proof. Put without loss of generality $\delta_n = \dfrac{1}{w(n)}$, where $w(1), w(2), \ldots$ denotes an increasing sequence of positive integers. Put $M_1 = 1$, $M_k = (k^2 + k^3 + \ldots + k^{w(k)+1})\, M_{k-1}$ $(k\geqq 2)$ and $N_k = w\,(M_1 + M_2 + \ldots + M_k) + 1$. (Other sequences $M_1, M_2, \ldots$ and $N_1, N_2, \ldots$ would do as well, but it is essential that $M_1, M_2, \ldots$ increase rapidly and $N_1, N_2, \ldots$ still more). Now for $k\geqq 1$ consider in $(0, 1)$ the set T_k consisting of N_k equidistant small segments $\sigma_k^i = \left(\dfrac{i}{N_k}, \dfrac{i}{N_k} + \dfrac{1}{N_k w(k)}\right)$, $i = 0, 1, \ldots, N_k - 1$. Let $f_k(x)$ denote the characteristic function of T_k, whereas $f(x)$ denotes the characteristic function of $T_1 + T_2 + \ldots$. Then

$$f(x) \leqq f_1(x) + f_2(x) + \ldots$$

is a function $\in L^2$ and $\int_0^1 f\,dx < \varepsilon$ by (6).

We now translate the numbers u_n In the first step we move the first M_1 elements of $u_1, u_2, \ldots$ In the second step the following M_2 elements etc.; hence after the k-th step $M_1 + \ldots + M_k$ elements have been moved. In the first step we move u_1 over a distance 0. Now let the $(k-1)$th step be carried out. Then we carry out the k-th step in substeps. In the first substep we remove the first $k^2 M_{k-1}$ elements $(n = M_1 + \ldots + M_{k-1} + 1, \ldots, M_1 + \ldots + M_{k-1} + k^2 M_{k-1})$. In the second step the following $k^3 M_{k-1}$ elements, etc. In the first substep we replace each u_n by an u'_n in such a way that $u'_n + \dfrac{1}{N_k w(k)}$ falls in the lefthand endpoint of a σ_k^i which is nearest to $u_n + \dfrac{1}{N_k w(k)}$ (mod 1).

In the h-th substep (denoted by (k, h)) we replace u_n by an u'_n in such a way that $u'_n + \frac{h}{N_k w(k)}$ falls in the lefthand endpoint of a σ_k^i which is nearest to $u_n + \frac{h}{N_k w(k)}$ (mod 1). Note that (mod 1) each u_n now is moved over a distance $< \frac{1}{N_k} \leqq \delta_k$. Now let x denote an arbitrary real number in (0, 1). Then x for each $k \geqq 2$ lies exactly in one of the $N_k w(k)$ equal parts of length $\frac{1}{N_k w(k)}$ in which we can divide the segment (0, 1), say in the part

$$\frac{h'}{N_k w(k)} \leqq x < \frac{h'+1}{N_k w(k)} \qquad (0 \leqq h' < N_k w(k)).$$

Now there is an uniquely defined integer $h = h(k)$ $(0 \leqq h < w(k))$ such that $h = h'$ (mod $w(k)$).

Consider the elements u_n, which have been moved by the substep (k, h). It is easily proved that the fractional part of the corresponding numbers $u'_n + x$ will belong to one of the segments σ_k^i. Hence $f(u'_n + x) = 1$. Denoting the total number of elements which have been moved after finishing the substep (k, h) by $A(k, h)$ we clearly find

$$\frac{1}{A(k, h)} \sum_{n=1}^{A(k, h)} f(u'_n + x) \geqq \frac{k^{h+1} M_{k-1}}{A(k, h)} \to 1 \text{ as } k \to \infty$$

by the definitions of M_{k-1} and $A(k, h)$. Q. e. d.

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