On the Gibbs' phenomenon for Euler means.

By O. SZÁSZ in Cincinnati.

1. Introduction.

Consider a Fourier sine series

(1.1) $$f(t) \sim \sum_{1}^{\infty} b_n \sin nt, \quad b_n = \frac{2}{\pi} \int_0^{\pi} f(t) \sin nt\, dt,$$

and its partial sums $s_n(t)$ $(n = 1, 2, 3, \ldots)$. FEJÉR proved[1]) that if $f(t)$ is of bounded variation, and if $nt_n \to \tau$ as $t_n \to +0$, then

$$s_n(t_n) \to \frac{2}{\pi} f(+0) \int_0^{\tau} \frac{\sin t}{t} dt \equiv \frac{2}{\pi} f(+0) J(\tau), \quad 0 \leqq \tau \leqq +\infty.$$

For $\tau = \pi$ $J(\tau)$ attains its maximal value,and

$$\lim_{nt_n \to \pi} s_n(t_n) = \frac{2}{\pi} f(+0) \int_0^{\pi} \frac{\sin t}{t} dt = f(+0) \times 1{,}1789797 \ldots .$$

Thus the limit-points of the partial sums as $t_n \to 0$ cover an interval which extends beyond $f(+0)$, if $f(+0) \neq 0$. This is called GIBBS' phenomenon, relative to the partial sums.

Our aim is to establish the corresponding phenomenon for Euler means.

For Cesàro means Gibbs' phenomenon was discussed by H. CRAMÉR and T. H. GRONWALL.

2. Euler means of the series $\Sigma \frac{\sin nt}{n}$.

The general Euler means of a sequence $\{s_n\}$ depend on a parameter r, and are defined by the triangular transform

$$\sigma_n(r) = \sigma_n = \sum_{\nu=0}^{n} \binom{n}{\nu} r^{\nu} (1-r)^{n-\nu} s_{\nu}, \quad n = 0, 1, 2, \ldots ;$$

We assume $0 < r \leqq 1$, in which case the summation method is regular.[2])

[1]) For references see A. ZYGMUND, *Trigonometrical series* (Warszawa, 1935); in particular pp. 179—181.

[2]) For $r = \frac{1}{2}$ it is essentially Euler's series transform; its regularity was first proved by L. D. AMES in 1901. Subsequent results are due to E. JACOBSTHAL, K. KNOPP and R. P. AGNEW.

Nevertheless the Euler means of a Fourier series may be divergent at a point of continuity of the function. Thus we may expect a Gibbs' phenomenon. We first consider the standard series $\sum_1^\infty \frac{\sin nt}{n} = \frac{1}{2}(\pi - t)$, and assume $0 < t < \pi$; now

$$s_0 = 0,\; s_n = \sum_1^n \frac{\sin \nu t}{\nu} = \int_0^t \left(\sum_1^n \cos \nu x\right) dx = -\frac{t}{2} + \int_0^t \frac{\sin (n+\frac{1}{2})x}{2 \sin \frac{1}{2} x}\, dx,$$

and

$$\sigma_n = -\frac{t}{2} + \int_0^t \frac{1}{2 \sin \frac{1}{2} x} \sum_0^n \binom{n}{\nu} r^\nu (1-r)^{n-\nu} \sin (\nu + \tfrac{1}{2})\, x\, dx.$$

The formula

$$\sum_0^n \binom{n}{\nu} r^\nu (1-r)^{n-\nu} e^{i\nu x} = (1 - r + re^{ix})^n$$

now yields

$$\sigma_n + \frac{t}{2} = \frac{1}{2} \Im \int_0^t \frac{1}{\sin \frac{1}{2} x} (1 - r + re^{ix})^n e^{ix/2} dx \text{ [3]}.$$

Let $1 - r + re^{ix} = \varrho e^{i\alpha}$, then

(2.1) $$\varrho \cos \alpha = 1 - r + r \cos x, \quad \varrho \sin \alpha = r \sin x,$$

(2.2) $$\varrho^2 = (1-r)^2 + r^2 + 2r(1-r) \cos x = 1 - 2r(1-r)(1 - \cos x) \leqq 1.$$

We assume $0 < x \leqq t \leqq \frac{\pi}{2}$; now

$$\sigma_n + \frac{t}{2} = \frac{1}{2} \int_0^t \frac{1}{\sin \frac{1}{2} x} \varrho^n \sin\left(n\alpha + \frac{x}{2}\right) dx = \frac{1}{2} \int_0^t \cot \frac{x}{2}\, \varrho^n \sin n\alpha\, dx + \frac{1}{2} \int_0^t \varrho^n \cos n\alpha\, dx.$$

Here

$$\left| \int_0^t \varrho^n \cos n\alpha\, dx \right| < t,$$

hence

$$\int_0^t \varrho^n \cos n\alpha\, dx = \eta t, \quad |\eta| < 1,$$

(2.3) $$\sigma_n + \frac{1-\eta}{2} t = \frac{1}{2} \int_0^t \varrho^n \cot \frac{x}{2} \sin n\alpha\, dx.$$

We now assume that

(2.4) $$t = t_n, \quad nt_n \to \tau, \; 0 \leqq \tau \leqq \infty, \quad nt_n^2 \to 0.$$

We have

$$0 < 1 - \varrho^n = (1 - \varrho) \sum_0^{n-1} \varrho^\nu < n(1 - \varrho),$$

[3]) $\Im$ means the imaginary part.

and from (2.2)

$$1-\varrho^2=4r(1-r)\sin^2 x/2<r(1-r)x^2,$$

so that

(2.5) $$1-\varrho<r(1-r)x^2\leqq x^2/4.$$

It follows that $1-\varrho^n<nx^2/4$, or $1-\varrho^n=\lambda n x^2$, $0<\lambda<1/4$, and

$$\int_0^t \varrho^n \cot\tfrac{1}{2}x \sin n\alpha\, dx=\int_0^t \cot\tfrac{1}{2}x \sin n\alpha\, dx-n\int_0^t \lambda x^2 \cot\tfrac{1}{2}x \sin n\alpha\, dx.$$

Now

$$n\left|\int_0^t \lambda x^2 \cot\tfrac{1}{2}x \sin n\alpha\, dx\right|<n\int_0^t x^2\cot\tfrac{1}{2}x\, dx<\tfrac{1}{3}\pi n t^2=o(1), \text{ as } n\to\infty,$$

in view of (2.4). Thus

(2.6) $$\int_0^t \varrho^n \cot\tfrac{1}{2}x \sin n\alpha\, dx=\int_0^t \cot\tfrac{1}{2}x \sin n\alpha\, dx+o(1).$$

Next, from (2.2) $\varrho^2\geqq(1-r)^2+r^2\geqq r^2$, $\varrho\geqq r$, and now from (2.1) $r\sin x= =\varrho\sin\alpha\geqq r\sin\alpha$, hence $\alpha<x$. It is well known that $0<x-\sin x<x^3$; now, from (2.1) $\varrho\alpha-rx=\varrho(\alpha-\sin\alpha)-r(x-\sin x)$, hence $|\varrho\alpha-rx|< <\alpha^3+x^3<2x^3$, and

$$|\alpha-rx|\leqq|\varrho\alpha-rx|+(1-\varrho)\alpha<2x^3+x^3=3x^3,$$

or $\alpha=rx+\mu x^3$, $|\mu|<3$. We now have

(2.7) $$\int_0^t \cot\tfrac{1}{2}x \sin n\alpha\, dx=\int_0^t \cot\tfrac{1}{2}x \sin nrx \cos n\mu x^3\, dx+$$

$$+\int_0^t \cot\tfrac{1}{2}x \cos nrx \sin n\mu x^3\, dx=I_1+I_2$$

and

(2.8) $$|I_2|<3n\int_0^t x^3\cot\tfrac{1}{2}x\, dx=O(nt^3)=o(1).$$

Finally

(2.9) $$I_1=\int_0^t \frac{\sin nrx}{x}dx-\int_0^t \sin nrx\left\{\frac{2}{x}-\frac{\cos\frac{1}{2}x\cos n\mu x^3}{\sin\frac{1}{2}x}\right\}dx=T_1(n)+T_2(n),$$

say, where

$$2\sin\tfrac{1}{2}x-x\cos\tfrac{1}{2}x\cos n\mu x^3=2\sin\tfrac{1}{2}x-x\cos\tfrac{1}{2}x+x\cos\tfrac{1}{2}x(1-\cos n\mu x^3).$$

From the mean value theorem $0\leqq 2\sin\frac{1}{2}x-x\cos\frac{1}{2}x<\frac{1}{4}x^3$ and

$$\int_0^t \sin nrx\frac{2\sin\frac{1}{2}x-x\cos\frac{1}{2}x}{x\sin\frac{1}{2}x}dx=o\left(\int_0^t x\, dx\right)=o(1).$$

Furthermore

$$\int_0^t \sin nrx \cot \tfrac{1}{2} x \,(1-\cos n\mu x^3)\, dx = O\left(\int_0^t n^2 x^5\, dx\right) = O(n^2 t^6) = o(1),$$

so that $T_2(n) \to 0$.

Collecting (2.3), (2.6), (2.7), (2.8) and (2.9), we find

$$2\sigma_n + (1-\eta)t_n = 2\int_0^{t_n} \frac{\sin nrx}{x}\, dx + o(1) = 2\int_0^{nrt_n} \frac{\sin y}{y}\, dy + o(1),$$

or

$$\sigma_n \to \int_0^{r\tau} \frac{\sin y}{y}\, dy, \text{ as } nt_n \to \tau, \quad 0 \leqq \tau \leqq \infty, \quad \text{and } nt_n^2 \to 0.$$

The complete discussion of the case $\limsup nt_n^2 > 0$ is more complicated, but we can prove in any case that $\limsup \sigma_n(t_n) \leqq \int_0^{\pi} \frac{\sin t}{t}\, dt$. This follows from

$$\limsup \sigma_n(t_n) \leqq \limsup s_n(t_n) \sum_0^n \binom{n}{\nu} r^\nu (1-r)^{n-\nu} = \int_0^{\pi} \frac{\sin t}{t}\, dt.$$

Summarizing, we have proved the following theorem:

Theorem 1. *For the Euler means of the series* $\sum_1^\infty \frac{\sin nt}{n}$ *we have*

$$\lim \sigma_n(t_n) = \int_0^{r\tau} \frac{\sin y}{y}\, dy \quad \textit{as} \quad nt_n \to \tau, \quad nt^2 \to 0,$$

and always

$$\limsup \sigma_n(t_n) \leqq \int_0^{\pi} \frac{\sin t}{t}\, dt.$$

3. Gibbs' phenomenon of Euler means for a class of Fourier series.

Suppose that the series (1.1) has a simple discontinuity at the point 0: $f(+0) = \pi A/2 > 0$, and let

(3.1) $\quad \varphi(t) = f(t) - A(\pi - t)/2 \sim \Sigma(b_n - An^{-1}) \sin nt \equiv \Sigma \beta_n \sin nt,$

so that $\varphi(t)$ is continuous at $t=0$. Suppose further that the series (3.1) is uniformly summable by Euler means at $t=0$. The behaviour of $\sigma_n\{f, t_n\}$ is then the same as for the series $A\sum' \frac{\sin nt}{n}$. In particular, if the series (3.1) is uniformly *convergent* at $t=0$, then its Euler means present the Gibbs' phenomenon exhibited in Theorem 1. A case in point is, when[4])

$$\lim_{\lambda \downarrow 1} \limsup_{n \to \infty} \sum_{n}^{\lambda n} (|b_\nu| - b_\nu) = 0. \tag{3.2}$$

Thus we have the theorem:

T h e o r e m 2. *If* (3.2) *holds, and if* $f(t)$ *has a simple discontinuity at* $t = 0$, *then the Euler means of the series* (1.1) *present the same Gibbs' phenomenon as in Theorem* 1.

F i n a l r e m a r k. We can discuss in a similar manner the Gibbs' phenomenon for certain Hausdorff means and for Borel summability.

University of Cincinnati,
National Bureau of Standards.

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[4]) O. Szász, On uniform convergence of Fourier series, *Bulletin American Math. Society*, **50** (1949), pp. 587—595; in particular Theorem 2.