On limit periodic functions of infinitely many variables.

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1. In the sequel the functions to be considered are continuous complex-valued functions of unrestricted real variables. Furthermore, convergence of a sequence of functions is always to be taken in the sense of uniform convergence over the whole range of the variable (or the variables).

2. Among the a.p. (almost periodic) functions of one variable x,

$$F(x) \sim \Sigma A_n e^{i\Lambda_n x},$$

the p.p. (purely periodic) functions $P(x)$ are the simplest ones; the periods of such a function are either all real numbers (in the case of $P(x)$ being constant), or the integral multiples of a real number $p_0 \neq 0$. Another simple, although more general case of a.p. functions $F(x)$ are the l.p. (limit periodic) functions $G(x)$ the set of which are obtained from the class of the p. p. functions $\{P(x)\}$ by closing it (with respect to uniform convergence), i. e.

$$\{G(x)\} = \mathrm{Cl}\,\{P(x)\}.$$

As easily seen (II, p. 141), for the l. p. functions $G(x)$ a kind of period still exists since two p.p. functions $P_1(x)$ and $P_2(x)$ which approximate a non-constant l.p. function $G(x)$ sufficiently well, must necessarily have periods with rational ratio. Denoting the class $\{G(x)\} = \mathrm{Cl}\,\{P(x)\}$ of all l. p. functions by C, and by $C_p\,(p \neq 0)$ the closure $\mathrm{Cl}\,\{P_p(x)\}$ of only those periodic functions $P_p(x)$ which have a rational multiple of p as one of their periods, we conclude that the set C is the union of all the sets C_p, i. e.

$$C = \bigcup_p C_p.$$

We may express this in the following way: We get the same set of functions (and not a smaller one) by closing first, for a fixed $p \neq 0$, the set $\{P_p(x)\}$ and then forming the union of all these closures, as we get by closing directly the whole set $\{P(x)\}$. As to the Fourier series of the l. p. functions, these are characterized, among the Fourier series of the a. p. functions, by having exponents Λ_n with mutually rational ratios; more particular, the exponents of a l. p. function from C_p are rational multiples of the number $\frac{2\pi}{p}$.

3. In the study of the a. p. functions of one variable the l. p. functions of an infinite number of variables play an important role (II, p. 118—163). We consider the enumerable-dimensional space of points $X=(x_1, x_2, \ldots)$ with arbitrary real coordinates where convergence of a sequence of points $X', X'', \ldots$ simply means convergence in each of the coordinates. A (continuous) function $P(X)=P(x_1, x_2, \ldots)$ is called p p. (purely periodic) with respect to the axis, if there exist non-vanishing real numbers $p_1, p_2, \ldots$ such that for each n the equation

$$P(x_1, x_2, \ldots, x_n+p_n, \ldots)=P(x_1, x_2, \ldots, x_n, \ldots)$$

holds good in the whole space. On account of the continuity of the function $P(X)$ we then also have (II, p. 135)

$$P(x_1+\nu_1 p_1, x_2+\nu_2 p_2, \ldots)=P(x_1, x_2, \ldots)$$

for each choice of the integers $\nu_1, \nu_2, \ldots$. By closing the set of all functions $P(X)$, p. p. with respect to the axis, we get the functions $G(X)$, l. p. with respect to the axis,

$$\{G(X)\}=\mathrm{Cl}\,\{P(X)\}.$$

Furthermore (II, p. 148), denoting for $p_1\neq 0, p_2\neq 0, \ldots$ by $C_{p_1, p_2, \ldots}$ the closure of only those of our p. p. functions $P_{p_1, p_2, \ldots}(X)$ which have rational multiples of $p_1, p_2, \ldots$ as periods with respect to the axis, we have just as before

$$C=\bigcup_{p_1, p_2, \ldots} C_{p_1, p_2, \ldots}.$$

Among the Fourier series of the a. p functions $F(X)$ of infinitely many variables

$$F(x_1, x_2, \ldots) \sim \Sigma A_n e^{i(\Lambda_{n,1}x_1+\Lambda_{n,2}x_2+\ldots+\Lambda_{n,m_n}x_{m_n})}$$

studied by BOCHNER (I), those belonging to l. p. functions of the class $C_{p_1, p_2, \ldots}$ are characterized by having, for each fixed m, all the numbers $\Lambda_{n,m}$ in the exponents equal to rational multiples of $\dfrac{2\pi}{p_m}$. Thus the Fourier series belonging to the l. p. functions of the class $C_{2\pi, 2\pi, \ldots}$ are just those Fourier series of a. p. functions for which the numbers $\Lambda_{n,m}$ are all rational.

4. We now introduce the notion of a substitution in our infinite-dimensional space as a linear one-to-one bicontinuous transformation T of the whole space on the whole space itself. As easily seen (III, p. 11 and V, p. 53) such a substitution may be written in the form

$$\begin{aligned} x_1 &= L_1(Y)=\alpha_{11}y_1+\alpha_{12}y_2+\ldots+\alpha_{1q_1}y_{q_1} \\ x_2 &= L_2(Y)=\alpha_{21}y_1+\alpha_{22}y_2+\ldots+\alpha_{2q_2}x_{q_2} \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

where the linear forms $L_m(Y)$, each containing only a finite number of the coordinates of Y, fulfill the following two conditions. 1^0. The $L_m(Y)$ are linearly independent, i. e. there exists no linear relation with constant, not all vanishing coefficients among any finite number of them. 2^0. Each variable y_q $(q=1, 2, \ldots)$ may be "isolated", i. e. expressed as a linear combination with constant coefficients of a finite number of the $L_m(Y)$. For a later application we remark that if a (finite or enumerable) set of linear forms satisfies only the condition 1^0 we may always (III, p. 14) add to the set new linear forms — and even such consisting each of only one coordinate — so that also condition 2^0 be fulfilled.

5. The notion of almost periodicity of a function $F(X)$ on our infinite-dimensional space is invariant under any substitution T performed on X, i. e. $F(TX)$ is again an a. p. function of X. In fact the set of the a. p functions $F(X)$ may be characterized as the closure of the set of the trigonometric polynomials $S(X)$, and a trigonometric polynomial $S(X)$ is evidently transformed into a trigonometric polynomial $S(TX)$. Now, applying a substitution T to a function $P(X)$, p. p. with respect to the axis, and denoting again the new variable point by X, instead of Y, we obtain a (continuous) function which we denote by $P_T(X)$ and abbreviatively call a p. p. function with respect to the substitution T (or more correctly with respect to the straight lines into which the coordinate axis are transformed, and which again span the whole space). For a fixed substitution T and all the $P(X)$ we form the class $\{P_T(X)\}$ and its closure $C_T = \mathrm{Cl}\{P_T(X)\}$, the functions of which we call l. p. functions with respect to T. Finally, we form the union Γ of all these classes C_T, $\Gamma = \bigcup_T C_T$, the functions of which we simply denote as l. p. functions.

6. Before proceeding, it may be illustrating to consider the notion of periodicity in our infinite-dimensional space from a more general point of view. A vector $V = (v_1, v_2, \ldots)$ in our space is called a period of the (continuous) function $F(X)$ if $F(X+V) = F(X)$ for all X. Each function has the trivial period $(0, 0, \ldots)$. Obviously, on account of the continuity of $F(X)$ the set of all periods of $F(X)$ is a closed module. Now, according to a simple, but not trivial theorem of E. FØLNER and myself (IV, p. 30 or V, p. 46) every closed module in our space may be transformed by a substitution T into a module of the simple type $\{(v_1, v_2, \ldots, v_n, \ldots)\}$ where the indices $1, 2, \ldots, n, \ldots$ fall into three classes $\{n_r\}$, $\{n_s\}$, $\{n_t\}$ such that the coordinates v_{n_r} independently run through all real numbers, the coordinates v_{n_s} independently run through all integers while the remaining coordinates v_{n_t} are all equal to 0. Here we are only interested in the case in which the last class $\{n_t\}$ is empty, as otherwise the module does not span the whole space (i. e. is lying in a proper subspace). Thus we see that there exists no other function $F(X)$ with a period module which span the whole space than those introduced above, i. e. functions belonging to one of the classes $\{P_T(X)\}$.

7. Returning to the l. p. functions, there exists in case of functions of infinitely many variables a problem which has no analogue for l. p. functions of one variable[1]) and to which B. JESSEN has called my attention, namely *whether the union Γ of all the closed sets $C_T = \mathrm{Cl}\{P_T(X)\}$ is identical with (or only forms a part of) the closure Γ^* of the whole set of all the p. p. functions.* Since Γ^* is closed and Γ obviously contains all the p. p. functions, the problem is, in other words, whether the set Γ is closed. The purpose of this paper is to give the solution of this problem by proving

Theorem. *The set $\Gamma = \bigcup_T C_T$ consisting of all the l. p. functions of infinitely many variables is a closed one.*

8. The proof to be given in the next section depends on the consideration of the Fourier series of the l. p. functions $G(X) = G(x_1, x_2, \ldots)$. From what have been said before it easily follows that a necessary and sufficient condition for a Fourier series of an a. p. function $F(X)$ to be that of one of our l. p. functions is that the linear forms in the exponents

$$M_n(X) = \Lambda_{n,1}x_1 + \Lambda_{n,2}x_2 + \ldots + \Lambda_{n,m_n}x_{m_n}$$

can be obtained from linear forms with mere rational coefficients by subjecting them to some linear substitution. From this characterization of the Fourier series of the l. p. functions we shall deduce the following

Lemma. *A necessary and sufficient condition for a Fourier series $\Sigma A_n e^{iM_n(X)}$ of an a. p. function to belong to an l. p. function is that in any relation which expresses one of the linear forms $M_n(X)$ as a linear combination of a finite number of linearly independent forms of the sequence $M_1(X), M_2(X), \ldots$, the occurring constant coefficients (uniquely determined) shall all be rational.*

That the condition is necessary can immediately be seen. In fact, as a linear substitution does not change linear relations, or linear independance, among the linear forms in the exponents, it suffices to prove that the condition is fulfilled for a function l. p. with respect to the axis, for instance of the special class $C_{2\pi, 2\pi, \ldots}$. But in this case all the occurring coefficients $\Lambda_{n,m}$ are rational numbers and hence the condition is evidently fulfilled since a finite number of ordinary linear equations with rational coefficients and only one solution can only have a solution in rational numbers.

In order to see that the condition is sufficient we proceed in the following manner. From the sequence $M_1(X), M_2(X), \ldots$ we first select (successively) a subsequence of which any finite number of its terms is linearly independent and such that any $M_n(X)$ may be expressed as a linear

[1]) The problem exists also for l. p. functions of a finite number of variables $x_1, x_2, \ldots, x_n$ $(n > 1)$ and the solution given below is also valid in this case. However, in the finite-dimensional case the problem may easily be solved without applying the theory of Fourier series.

combination of a finite number of forms of this subsequence. To the linear forms of this subsequence we may, as remarked before, add new linear forms such that the enlarged (enumerable) set of linear forms may be used as right-hand sides of a linear substitution. By performing this substitution — or rather the inverse one — on our Fourier series $\Sigma A_n e^{iM_n(X)}$ we evidently obtain, from the assumption that the condition be fulfilled, a new Fourier series with mere rational coefficients in the exponents. Thus the corresponding function and hence also the function $F(X)$ before the transformation is a l.p. function.

9. We can now easily prove our theorem, viz. that the set Γ of all l. p. functions $G(X)$ is closed. We have to prove that if $F(X) \sim \Sigma A_n e^{iM_n(X)}$ is an a. p., but not a l.p. function, then $F(X)$ cannot be approximated uniformly by l. p functions. According to our lemma there exists among the linear forms $M_n(X)$ in the exponent of the Fourier series of $F(X)$ a linear relation

$$M_N(X) = b_1 M_{n_1}(X) + b_2 M_{n_2}(X) + \ldots + b_s M_{n_s}(X)$$

with linearly independent $M_{n_1}(X), M_{n_2}(X), \ldots, M_{n_s}(X)$ and not all b's rational. Now, as well-known, uniform convergence of a sequence of a. p. functions towards an (a. p.) function $F(X)$ implies formal convergence of the Fourier series of the functions of the sequence towards the Fourier series $\Sigma A_n e^{iM_n(X)}$ of $F(X)$. Hence in the Fourier series of any a. p. function $H(X)$ which approximates $F(X)$ sufficiently close each of the finite number of linear forms $M_N(X), M_{n_1}(X), \ldots, M_{n_s}(X)$ must necessarily occur as exponents, simply because they occur in the Fourier series of $F(X)$. Consequently, using the lemma once more, we see that the a. p. function $H(X)$ cannot be l. p. This proves the theorem.

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