Commutativity and spectral properties of normal operators.

By PAUL R. HALMOS in Chicago.

1. The results of this note grew out of a current investigation of spectral properties of operators on Hilbert space. While the characterization of the spectral manifolds of a normal operator (Theorem 1) appears to be new and may be considered to be of independent interest, I present it here mainly because it supplies an extremely easy proof of a theorem (Theorem 2) which was unknown until a few weeks ago. J. VON NEUMANN has asked whether or not it is true that if an operator B commutes with a normal operator A, then B commutes with A^* also. Well known and quite elementary considerations show that in order to answer the question affirmatively it is sufficient to prove that, under the stated hypotheses, B is reduced by all the spectral manifolds of A. This has recently been proved by B. FUGLEDE — he communicated his proof to me at the Boulder meeting of the American Mathematical Society at the end of August, 1949. The proof I present below is somewhat different from his in spirit and in method. I should say also that FUGLEDE's proof is valid for not necessarily bounded transformations A and that, similarly, only minor modifications are needed to adapt my proof to this more general case.

For the orientation of the reader I present here the trivial proof of the theorem under discussion for the case in which A has pure point spectrum; the proof of the general theorem below uses essentially the same idea and method. If λ is a proper value of A and if $\mathfrak{F}$ is the subspace of all corresponding proper vectors, then the relations $A(Bx) = B(Ax) = B(\lambda x) = \lambda(Bx)$ show that $\mathfrak{F}$ is invariant under B. Since to say that A has pure point spectrum means that the entire Hilbert space is spanned by orthogonal subspaces such as $\mathfrak{F}$, it follows that the orthogonal complement of $\mathfrak{F}$ is also invariant under B, and this is exactly what was to be proved.

2. Throughout this note I shall deal with a fixed complex Hilbert space $\mathfrak{H}$. An *operator* is a bounded linear transformation of $\mathfrak{H}$ into itself; an operator A is *normal* if it commutes with its adjoint A^*. If A is normal, then

$\|Ax\|^2=(Ax, Ax)=(A^*Ax, x)=(AA^*x, x)=(A^*x, A^*x)=\|A^*x\|^2$ for every vector x; it is easy to see that the identity $\|Ax\|=\|A^*x\|$ is not only necessary but also sufficient for the normality of A. A *subspace* is a closed linear manifold in $\mathfrak{H}$; a subspace $\mathfrak{M}$ *reduces* an operator A if both $\mathfrak{M}$ and $\mathfrak{M}^\perp$ (= the orthogonal complement of $\mathfrak{M}$) are invariant under A, i. e. if $A\mathfrak{M}\subset\mathfrak{M}$ and $A\mathfrak{M}^\perp\subset\mathfrak{M}^\perp$. There are two useful and elementary necessary and sufficient conditions that a subspace $\mathfrak{M}$ reduce an operator A; the first is that $\mathfrak{M}$ be invariant under both A and A^*, and the second is that A commute with the projection on $\mathfrak{M}$.

Lemma 1.[1]) *If A is a normal operator and if $\mathfrak{F}(A)=\{x:\ \|A^n x\|\leqq\|x\|,$ $n=1,2,\dots\}$, then $\mathfrak{F}(A)$ is a subspace and $\mathfrak{F}(A)$ is invariant under every operator B which commutes with A.*

Proof. Write $\mathfrak{S}$ for the set of all those vectors x for which the sequence $\{\|A^n x\|: n=1,2,\dots\}$ is bounded. Since $\|A^n(\alpha x)\|=|\alpha|\cdot\|A^n x\|$ and $\|A^n(x+y)\|\leqq\|A^n x\|+\|A^n y\|$, it follows that $\mathfrak{S}$ is a linear manifold; if an operator B commutes with A, then the relation $\|A^n(Bx)\|=\|B(A^n x)\|\leqq$ $\leqq\|B\|\cdot\|A^n x\|$ implies that $\mathfrak{S}$ is invariant under B. Clearly $\mathfrak{F}(A)$ is a closed set and $\mathfrak{F}(A)\subset\mathfrak{S}$; the proof of the lemma will be completed by showing that $\mathfrak{F}(A)=\mathfrak{S}$. For this purpose it is sufficient to show that if x is a vector such that, for some positive integer p, $\|A^p x\|>\alpha\|x\|$, $\alpha>1$, then the sequence $\{\|A^n x\|\}$ cannot be bounded. Since $\alpha^2\|x\|^2<\|A^p x\|^2=(A^p x, A^p x)=(A^{*p}A^p x, x)\leqq$ $\leqq\|A^{*p}A^p x\|\cdot\|x\|=\|A^{2p}x\|\cdot\|x\|$, it follows that $\|A^{2p}x\|>\alpha^2\|x\|$. Since an inductive repetition of this argument shows that $\|A^{2^k}x\|>\alpha^{2^k}\|x\|$ for every positive integer k, the proof is complete.

3. A *spectral measure* is a function E from the class of all Borel subsets of the set Δ of all complex numbers to projections on $\mathfrak{H}$, such that (i) $E(\Delta)=1$, (ii) $E(M\cap N)=E(M)E(N)$ whenever M and N are Borel sets, and (iii) $E(M)=\sum_{j=1}^{\infty}E(M_j)$ whenever $\{M_j\}$ is a disjoint sequence of Borel sets whose union is M (the series being understood to converge in the strong topology of operators).

Lemma 2. *If E is a spectral measure and if $\mathfrak{E}(M)=\{x:\ E(M)x=x\}$ for every Borel set M, then $\mathfrak{E}(M)$ is the subspace spanned by the class of all subspaces of the form $\mathfrak{E}(N)$, where N is an arbitrary compact subset of M.*

Proof. The assertion of the theorem is that, in a sense well known in the theory of numerical measures, every spectral measure is regular. The proof may be given along lines entirely similar to the numerical case, or it

[1]) This lemma is proved for Hermitian operators by B. A. Lengyel and M. H. Stone, Elementary proof of the spectral theorem, *Annals of Math.*, **37** (1936), pp. 853–864; cf. in particular p. 858. The following proof is a slight simplification of their proof.

may be reduced to that case as follows. All that it is necessary to prove is that if x is a vector in $\mathfrak{E}(M)$ such that x is orthogonal to $\mathfrak{E}(N)$ for every compact subset N of M, then $x=0$. Since, however, by the regularity of numerical measures, $\|x\|^2=\|E(M)x\|^2=\sup_N \|E(N)x\|^2$, it follows that there exists a countable class $\{N_j\}$ of compact subsets of M such that $\|x\|^2=\sup_j \|E(N_j)x\|^2$, and hence that indeed $x=0$.

I shall make use below of the spectral theorem for normal operators in the following form. If A is a normal operator, then there exists a unique spectral measure E, called the spectral measure of A, such that $(Ax,y)=$ $=\int \lambda\, d(E(\lambda)x,y)$ for every pair of vectors x and y.

4. In this final section I shall assume that A is a fixed normal operator with spectral measure E. For every complex number λ and every positive real number ε, I shall write $\mathfrak{F}(\lambda,\varepsilon)$ for $\mathfrak{F}\left(\frac{A-\lambda}{\varepsilon}\right)$; for every set M of complex numbers and every positive real number ε, I shall write $\mathfrak{F}(M,\varepsilon)$ for the subspace spanned by all those $\mathfrak{F}(\lambda,\varepsilon)$ for which $\lambda\in M$; and, for every set M of complex numbers, I shall write $\mathfrak{F}(M)=\bigcap_{\varepsilon>0}\mathfrak{F}(M,\varepsilon)$. Let $F(\lambda,\varepsilon)$, $F(M,\varepsilon)$, and $F(M)$ be the projections on the subspace $\mathfrak{F}(\lambda,\varepsilon)$, $\mathfrak{F}(M,\varepsilon)$, and $\mathfrak{F}(M)$, respectively.

Theorem 1. *For every compact set M, $\mathfrak{F}(M)=\mathfrak{E}(M)$.*

Proof. For any positive number ε, let $\{M_j\}$ be a disjoint sequence of non empty Borel sets of diameter not greater than ε and such that $\bigcup_j M_j=M$. If $x\in\mathfrak{E}(M)$, $x_j=E(M_j)x$, and $\lambda_j\in M_j$, then $\|(A-\lambda_j)^n x_j\|^2=$ $=\int_{M_j}|(\lambda-\lambda_j)^n|^2\, d(E(\lambda)x_j,x_j)\leqq\varepsilon^{2n}\|x_j\|^2$, so that, for each j, $x_j\in\mathfrak{F}(\lambda_j,\varepsilon)\subset\mathfrak{F}(M,\varepsilon)$. Since $x=E(M)x=\Sigma_j E(M_j)x=\Sigma_j x_j$, it follows that $x\in\mathfrak{F}(M,\varepsilon)$. The arbitrariness of ε implies that $x\in\mathfrak{F}(M)$, and the arbitrariness of x implies, consequently, that $\mathfrak{E}(M)\subset\mathfrak{F}(M)$. Note that this argument did not make use of compactness of M.

Suppose now that N is a compact subset of $A-M$, and let δ be the distance between M and N. If $\lambda_0\in M$, if $0<\varepsilon<\delta$, and if $x\in\mathfrak{F}(\lambda_0,\varepsilon)$, then $\|(A-\lambda_0)^n x\|\leqq\varepsilon^n\|x\|$; if, on the other hand, $x\in\mathfrak{E}(N)$, then $\|(A-\lambda_0)^n x\|^2=$ $=\int_N|(\lambda-\lambda_0)^n|^2\, d(E(\lambda)x,x)\geqq\delta^{2n}\|x\|^2$. It follows that $\mathfrak{F}(\lambda_0,\varepsilon)\cap\mathfrak{E}(N)=\{0\}$. Since $E(N)$ commutes with A, it follows from Lemma 1 that $\mathfrak{F}(\lambda_0,\varepsilon)$ is invariant under $E(N)$ and hence, since $E(N)$ is Hermitian, that $E(N)$ commutes with $F(\lambda_0,\varepsilon)$. This in turn implies that $F(\lambda_0,\varepsilon)E(N)$ is the projection on $\mathfrak{F}(\lambda_0,\varepsilon)\cap\mathfrak{E}(N)$, i. e. that $F(\lambda_0,\varepsilon)E(N)=0$, and it follows that $\mathfrak{F}(\lambda_0,\varepsilon)$ is orthogonal to $\mathfrak{E}(N)$. The validity of this assertion for every λ_0 in M shows that $\mathfrak{F}(M, \varepsilon)$ is orthogonal to $\mathfrak{E}(N)$ and therefore, *a fortiori*, that $\mathfrak{F}(M)$ is orthogonal to $\mathfrak{E}(N)$.

The result of the preceding paragraph implies, in view of Lemma 2, that $\mathfrak{F}(M)$ is orthogonal to $\mathfrak{E}(\Lambda - M)$. This means that $\mathfrak{F}(M) \subset (\mathfrak{E}(\Lambda - M))^{\perp} = \mathfrak{E}(M)$, and the proof of the theorem is complete. I remark that it is easy to construct examples to show that if M is not compact, then $\mathfrak{E}(M)$ may be a proper subset of $\mathfrak{F}(M)$.

Theorem 2. *If an operator B commutes with A, then $\mathfrak{E}(M)$ reduces B for every Borel set M.*

Proof. It follows from Lemma 1 that, for every complex number λ and every positive number ε, $\mathfrak{F}(\lambda, \varepsilon)$ is invariant under B, and hence that $\mathfrak{F}(M, \varepsilon)$ and $\mathfrak{F}(M)$ are invariant under B for every set M. Theorem 1 implies that $\mathfrak{E}(M)$ is invariant under B whenever M is compact and hence, by Lemma 2, that $\mathfrak{E}(M)$ is invariant under B for every Borel set M. Since $(\mathfrak{E}(M))^{\perp} = \mathfrak{E}(\Lambda - M)$, it follows automatically that $(\mathfrak{E}(M))^{\perp}$ is also invariant under B and hence that $\mathfrak{E}(M)$ reduces B.

University of Chicago.

(Received October 13, 1949.)