A new proof of the general ergodic theorem.

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The purpose of this note is to give a short and elementary proof of HUREWICZ's ergodic theorem [5] (the ergodic theorem without invariant measure). Our proof is a modification of a proof of BIRKHOFF's ergodic theorem [1] given by R. SALEM in his course at M. I. T. SALEM's proof, as well as a proof given by E. HOPF [4], is in turn a modification of one given by H. R. PITT. [8]. All the above mentioned proofs are closely related to RIESZ's proof [9] of BIRKHOFF's ergodic theorem. With the aid of the same method we shall also prove a generalization of HUREWICZ's theorem for transformations which are single valued but not necessarily one to one.

Let $(S, \mathfrak{B}, m)$ be a measure space where S is a set of elements called points and denoted by $x, y, \ldots, \mathfrak{B}$ a Borel field of subsets of S and m a countably additive non-negative set function defined for the sets belonging to $\mathfrak{B}$. The sets belonging to $\mathfrak{B}$ are called measurable sets and the set function m is called a measure. We assume that $S \in \mathfrak{B}$ and that S is a union of a countable number of measurable sets of finite measure.

Let T be a one-to-one point transformation of S onto itself. We shall say that T is *measurable* if both T and T^{-1} transform measurable sets into measurable sets. Thus if T is measurable so is T^n for $n = 0, \pm, \pm 2, \ldots$. We shall say that T is positively (negatively) *non-singular* if $T\,(T^{-1})$ transforms sets of measure zero into sets of measure zero. Thus if T is positively non-singular, so is T^n for $n = 2, 3, \ldots$.

Consider now the set functions m_n defined by $m_n(A) = m(T^n A)$, where $A \in \mathfrak{B}$ and $n = 0, 1, 2, \ldots$. If T is measurable and one-to-one, m_n is a countably additive non-negative set function defined for all sets A belonging to $\mathfrak{B}$. If, moreover T^n is positively non-singular, m_n is absolutely continuous with respect to m for $n = 0, 1, 2, \ldots$. It follows then by the Radon—Nikodym theorem that there exists a measurable function $w_n(x)$ such that for every $A \in \mathfrak{B}$ we have $m_n(A) = \int_A w_n(x)\, dm$, $n = 0, 1, 2, \ldots$. By considering approximating sums to the integrals in question one can show that

$$\int_S f(x)\, dm = \int_S f(T^n x)\, w_n(x)\, dm \tag{1}$$

for $n = 0, 1, 2, \ldots$, for any measurable set A and for any measurable function

$f(x)$ such that either its positive or negative part is integrable. For any $A \in \mathfrak{B}$ consider $m_{i+j}(A) = m(T^{i+j}A) = \int_A w_{i+j}(x)\,dm$. Also

$$m(T^{i+j}A) = m_i(T^jA) = \int_{T^jA} w_i(x)\,dm = \int_A w_i(T^jx)\,w\ (x)\,dm.$$

It follows that

$$(2) \qquad w_{i+j}(x) = w_i(T^jx)\,w_j(x) \quad \text{almost everywhere on } S,$$

and it can be assumed with no loss of generality that the equality in (2) holds everywhere on S, for any $i, j = 0, 1, 2, \ldots$.

Consider now any integrable real-valued function $q(x)$ and let

$$q^n(x) = q(x)\,w_0(x) + q(Tx)\,w_1(x) + \ldots + q(T^{n-1}x)\,w_{n-1}(x) = \sum_{i=0}^{n-1} q(T^ix)\,w_i(x).$$

We shall now state and prove HUREWICZ's ergodic theorem in a form given to it by HALMOS [2]. (For the relation between HUREWICZ's theorem and Theorem 1, see [7].)

Theorem I. *If T is a measurable, positively non-singular, one-to-one transformation of S onto itself, if $f(x)$ is integrable and if $h(x)$ is non-negative and such that* $\lim h^n(x) = \infty$ *almost everywhere, then $f^n(x)/h^n(x)$ converges almost everywhere to a finite limit.*

Remark. Our proof of theorem I. like that of HALMOS [2] and the corresponding proofs of HUREWICZ [5], KHINTCHINE [6] and HOPF [3], depends essentially on the following inequality:

Lemma 1. *Let $q(x)$ be any measurable function such that either its positive or negative part is integrable. Let E be the set of points x such that $q^n(x) \geqq 0$ for some n. Then* $\int_E q(x)\,dm \geqq 0$

In fact, the difference between our proof and those mentioned above lies essentially only in the proof of the inequality. We shall therefore restrict ourselves to the proof of Lemma 1.

Lemma 2. *Let $u_0, u_1, \ldots$ be an infinite sequence of real numbers, and N a fixed positive integer. Suppose that*

$$\max_{1 \leqq n \leqq N} \sum_{i=0}^{n-1} u_{i+j} \geqq 0 \qquad \textit{for all } j \geqq 0.$$

Then

$$\sum_{i=0}^{\nu-1} u_i + \sum_{i=\nu}^{\nu+N-1} (u_i)^+ \geqq 0 \qquad \textit{for all } \nu \geqq 1,$$

where $(u_i)^+ = \max(u_i, 0)$.

Proof of the Lemma 2. By assumption, there exists an increasing sequence of integers $n_0 = 0, n_1, n_2, \ldots$ such that $0 < n_k - n_{k-1} \leqq N$, $u_{n_{k-1}} + \ldots + u_{n_k - 1} \geqq 0$, $k = 1, 2, \ldots$. For any ν, let p be such that $n_{p-1} < \nu \leqq n_p$.

Then $n_p < \nu + N$ and consequently

$$\sum_{i=0}^{\nu-1} u_i + \sum_{i=\nu}^{\nu+N-1} (u_i)^+ \geqq \sum_{i=0}^{n_p-1} u_i + \sum_{i=n_p}^{\nu+N-1} (u_i)^+ \geqq \sum_{k=1}^{p} (u_{n_{k-1}} + \dots + u_{n_k-1}) + \sum_{i=n_p}^{\nu+N-1} (u_i)^+ \geqq 0.$$

Proof of Lemma 1. Let $q(x)$ be any measurable function such that either its positive or negative part is integrable. Fix a positive integer N, and let E_N be the set of points x where $q^n(x) \geqq 0$ for some n, $1 \leqq n \leqq N$. Since it is clear that $E_N \subset E_{N+1}$ and that the union of all the sets E_N $(N = 1, 2, \dots)$ is equal to E, it suffices to prove that

$$\int_{E_N} q(x)\, dm \geqq 0.$$

Let us put $g(x) = q(x)$ if $x \in E_N$ and $g(x) = 0$ if $x \in S - E_N$. We first notice that $\int_S g(x)\, dm = \int_{E_N} q(x)\, dm$. Thus it suffices to show that $\int_S g(x)\, dm \geqq 0$. Next we notice that $g(x) \geqq q(x)$ for all $x \in S$. This is clear if $x \in E_N$, and $g(x) = 0 > \operatorname*{Max}_{1 \leqq n \leqq N} q^n(x) \geqq q^1(x) = q(x)$ if $x \in S - E_N$. From this follows that

$$\operatorname*{Max}_{1 \leqq n \leqq N} \sum_{i=0}^{n-1} g(T^i x)\, w_i(x) \equiv \operatorname*{Max}_{1 \leqq n \leqq N} g^n(x) \geqq 0 \text{ for all } x \in S. \tag{3}$$

In fact, $\operatorname*{Max}_{1 \leqq n \leqq N} g^n(x) \geqq \operatorname*{Max}_{1 \leqq n \leqq N} q^n(x) \geqq 0$ if $x \in E_N$ and $\operatorname*{Max}_{1 \leqq n \leqq N} g^n(x) \geqq g^1(x) = g(x) = 0$ if $x \in S - E_N$.

If we replace x by $T^j(x)$ in (3) and multiply both sides by $w_j(x)$, we obtain from (3) and (2) that

$$\operatorname*{Max}_{1 \leqq n \leqq N} \sum_{i=0}^{n-1} g(T^{i+j} x)\, w_{i+j}(x) \geqq 0 \quad \text{for all } x \in S \text{ and all } j \geqq 0.$$

We can thus apply the lemma to the sequence $u_i = g(T^i x)\, w_i(x)$ and obtain

$$G_\nu(x) \equiv \sum_{i=0}^{\nu-1} g(T^i x)\, w_i(x) + \sum_{i=\nu}^{\nu+N-1} (g(T^i x))^+ w_i(x) \geqq 0$$

for all $x \in S$ and all $\nu \geqq 1$. Hence $\int_S G_\nu(x)\, dm \geqq 0$. But from (1) we see that $\int_S G_\nu(x)\, dm = \nu \int_S g(x)\, dm + N \int_S (g(x))^+ dm$. Hence

$$\int_S g(x)\, dm + \frac{N}{\nu} \int_S (g(x))^+ dm \geqq 0.$$

Now if $(g(x))^+$ is integrable, we see by letting ν tend to ∞ that $\int_S g(x)\, dm \geqq 0$. If $(g(x))^+$ is not integrable, then $(g(x))^- = \operatorname{Max}(-g(x), 0)$ is integrable and $\int_S g(x)\, dm = \infty > 0$. (Notice that while in the course of the proof we have used the fact that T is only positively non-singular, it is true that the assumption that T is so together with the assumption that $\lim h^n(x) = \infty$ almost everywhere implies that T is also negatively non-singular.)

We turn our attention now to transformations which are single-valued but not necessarily one-to-one. We will state and prove a generalization of Theorem I. Let $(S, \mathfrak{B}, m)$ be a measure space and let T be a single-valued transformation of S onto itself. We assume that T is measurable, i. e that both TA and $T^{-1}A$ are measurable if A is measurable. We also assume that T is positively non-singular, i. e. that $m(T^{-1}A) = 0$ implies that $m(A) = 0$. Consider the set functions $m_n(A) = m(T^n A)$ for $n = 1, 2, \ldots$ and for every measurable A. We see immediately that $m_n(A)$ is not necessarily additive and hence is not a measure on $(S, \mathfrak{B})$. Thus the procedure for defining the weight functions $w_n(x)$ cannot be followed here as previously and has to be modified. In this modification we are governed by the fact that our proposed theorem must reduce to the known special cases, i. e. to Theorem I in case T is one-to-one and to BIRKHOFF's theorem in case T is measure preserving in the sense that $m(T^{-1}A) = m(A)$. (Cf. F. RIESZ [9].) In fact let $\mathfrak{B}_1$ be the collection of all sets which are full inverse images of sets belonging to $\mathfrak{B}$. It is quite easy to see that $\mathfrak{B}_1$ is a Borel field of sets. It is also quite easy to see that $m_1(A) = m(TA)$ is a completely additive set function on $\mathfrak{B}_1$ and thus both m and m_1 are measures defined on $(S, \mathfrak{B}_1)$. Moreover m_1 is absolutely continuous with respect to m on $(S, \mathfrak{B}_1)$. Thus by using the RADON—NIKODYM theorem we see that there exists a $\mathfrak{B}_1$-measurable point function $w_1(x)$ such that

$$m_1(A) = \int_A w_1(x)\, dm,$$

for any $A \in \mathfrak{B}_1$. $w_1(x)$ is positive almost everywhere and without loss of generality we can assume that $w_1(x)$ is positive everywhere.

Let us now define $w_n(x) = w(T^{n-1}x) \ldots w(Tx)\, w(x)$, $n = 2, 3, \ldots$. By considering approximating sums to the integrals in question one can see that

$$\int_S f(x)\, dm = \int_S f(Tx)\, w_1(x)\, dm$$

for every measurable function $f(x)$ such that either its positive or negative part is integrable. It follows that

$$(1') \qquad \int_S f(x)\, dm = \int_S f(T^n x)\, w_n(x)\, dm$$

for $n = 0, 1, 2, \ldots$ and for every $f(x)$ which is described above. By definition we have

$$(2') \qquad w_{i+j}(x) = w_i(T^j x)\, w_j(x) \quad \text{for} \quad i, j = 0, 1, \ldots.$$

With $w_n(x)$ as weight functions we form for every $\mathfrak{B}$-measurable function $q(x)$ the sum

$$q^n(x) = q(x) + q(Tx)\, w_1(x) + \ldots + q(T^{n-1}x)\, w_{n-1}(x).$$

Theorem II. *If T is a single valued measurable and non-singular point transformation of $(S, \mathfrak{B}, m)$ onto itself, if $f(x)$ is integrable and if $h(x)$ is non-negative and such that $h^n(x) \to \infty$ almost everywhere, then $f^n(x)/h^n(x)$ converges almost everywhere to a finite limit.*

The proof of Theorem II follows exactly the same lines as that of Theorem I and we shall therefore omit it here.

The question now arises as to when is it true that $h^n(x) \to \infty$ almost everywhere if, for instance, $h(x)$ is positive almost everywhere. In case T is one-to-one it was shown by HALMOS [2] (p. 157) that for $h(x) > 0$ almost everywhere, $h^n(x) \to \infty$ almost everywhere if there are no wandering sets of positive measure with respect to T, i. e. $T^i A \cap A = 0$ for $i = \pm 1, \pm 2, \ldots$ implies $m(A) = 0$. Thus the condition that $h^n(x) \to \infty$ can, at least for the case of a one-to-one transformation, be replaced by a condition which reflects directly on the nature of the transformation. In the more general case of a single valued transformation which is not one-to-one we have not been able to replace the condition $h^n(x) \to \infty$ by one directly bearing on the nature of T. We have been able to show that if there exists a measure μ on $(S, \mathfrak{B})$ which is invariant under T $(\mu(T^{-1}A) = \mu(A))$ and if T admits no wandering sets of positive measure then if $h(x) > 0$ almost everywhere, we have $h^n(x) \to \infty$. But in general the question of whether the condition that there are no wandering sets of positive measure under T (or some similar condition) yields $h^n(x) \to \infty$ for $h(x) > 0$ almost everywhere is still open.

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