The asymptotic behaviour of the coefficients of certain power series.

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1. The purpose of this paper is to derive an asymptotic expansion (as $k\to\infty$) for the coefficient $A_k = A_k(K, \alpha)$ in

$$g(z) = e^{K[z-f(\alpha z)]} = \sum_{\mu=0}^{\infty} \frac{1}{\mu!} A_\mu z^\mu, \quad K = \frac{1}{\sigma} k + \tau, \tag{1}$$

where σ, τ are real constants, $\sigma > 0$, and $f(z)$ is analytic at $z=0$.

Under certain assumptions on $f(z)$ one can easily obtain a principal term for A_k by a method introduced by P. DEBYE[1]), called the method of steepest descent (Sattelpunktmethode). Using the theorem of residues we have

$$\frac{1}{k!} A_k = \frac{1}{2\pi i} \int \frac{1}{z} e^{\left(\frac{1}{\sigma} k+\tau\right)[z-f(\alpha z)] - k \log z} dz$$

taken over a small circle surrounding the origin. By CAUCHY's theorem we can displace the contour so that it shall pass through the "saddle point" z_0 at which $\frac{d}{dz}\left(\frac{1}{\sigma}(z-f(\alpha z)) - \log z\right) = 0$ or, $z_0 = \sigma + \alpha z_0 f'(\alpha z_0)$. It can be shown that if the contour passes through z_0 in a suitable direction and $f(z)$ behaves appropriately on the rest of the path of integration, then the chief contribution to the integral is furnished by the neighbourhood of z_0 and has the form

$$(2\pi c k)^{-\frac{1}{2}} e^{\left(\frac{1}{\sigma} k+\tau\right)[z_0 - f(\alpha z_0)] - k \log z_0} \tag{2}$$

for a certain constant c which depends on the particular conditions of the problem. The difficulty of the method lies in the proper choice of the path of integration which has to be determined individually for each function $f(z)$. If we cannot make sure that the contribution due to the complementary part of the path is negligible then the method has a rather heuristic value.

In the present paper I shall obtain a full asymptotic expansion for A_k by an entirely different method which avoids complex integration altogether.

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[1]) See e. g. G. SZEGŐ, *Orthogonal polynomials* (New York, 1939), p. 215 ff. I am indebted to P. TURÁN for having called my attention to this method.

The method has a more or less formal elementary character and does not require a knowledge of the functiontheoretical properties of $f(z)$ except that it has a power series at $z=0$. In fact, I shall prove the following

Theorem 1. *Let* $f(z)=\sum_{\nu=1}^{\infty}\frac{1}{\nu}c_\nu z^\nu$ *be an arbitrary power series with positive radius of convergence and constant term zero*[2]*;* A_k *shall be defined as above. Let* $u=u(t)=\sum_{\nu=0}^{\infty}d_\nu t^{\nu+1}$, $d_0=\sigma$, *denote the inverse function of*

$$t=\frac{u}{\sigma+uf'(u)},\quad u=t\left(\sigma+\sum_{\nu=1}^{\infty}c_\nu u^\nu\right) \tag{3}$$

and let $v=v(t)=\sum_{\nu=0}^{\infty}D_\nu t^{\nu+1}$, $D_0=\sigma$ *be defined by*

$$v=t\left(\sigma+\sum_{\nu=1}^{\infty}|c_\nu|v^\nu\right). \tag{4}$$

Then

$$A_k=K^k\left(\frac{u(\alpha)}{\sigma\alpha}\right)^{\sigma\tau-1}\left(\frac{1}{\sigma}u'(\alpha)\right)^{\frac{1}{2}}\exp\left\{-K\int_0^\alpha\frac{u(t)-\sigma t}{t^2}dt\right\}\left(1+\sum_{\mu=1}^{m}k^{-\mu}\psi_\mu(\alpha)+O(k^{-m-1})\right) \tag{5}$$

for every $m\geq 0$ *and certain functions* $\psi_\mu(\alpha)$, *analytic in* $|\alpha|<\varrho$, *where* ϱ *denotes the radius of convergence of* $v(t)$. *The expansion (5) is uniformly valid for* $|\alpha|\leq\varrho_0<\varrho$.

The constant in $O(k^{-m-1})$ depends, of course, on ϱ_0 and on m and a similar remark pertains to all O-notations in the paper. $\left(\frac{u}{\sigma\alpha}\right)^{\sigma\tau-1}$ and $\left(\frac{1}{\sigma}u'\right)^{\frac{1}{2}}$ denote the principal branches of the functions, which have the value 1 at $\alpha=0$. No ambiguity is involved in $\int_0^\alpha\frac{u-\sigma t}{t^2}dt$ for complex values of α since the integrand is obviously regular for $|t|\leq|\alpha|$.

The range of validity of (5) is not necessarily confined to the circle with radius ϱ. In fact, it might be possible to "continue" the expansion into new regions of the complex plane by varying the coefficient c_1. It is quite possible that (5) always holds in the interior of the circle of convergence of $u(t)$.

Using Stirling's formula, we obtain from (5) for the coefficients of the power series (1)

$$\frac{1}{k!}A_k=(2\pi)^{-\frac{1}{2}}e^{\sigma\tau}\left(\frac{u(\alpha)}{\sigma\alpha}\right)^{\sigma\tau-1}\left(\frac{1}{\sigma}u'(\alpha)\right)^{\frac{1}{2}}k^{-\frac{1}{2}}\left(\frac{e}{\sigma}\right)^k\times \times\exp\left\{-K\int_0^\alpha\frac{u-\sigma t}{t^2}dt\right\}\left(1+\sum_{\mu=1}^{m}k^{-\mu}\lambda_\mu(\alpha)+O(k^{-m-1})\right) \tag{6}$$

[2]) Obviously this last assumption can be made without loss of generality.

where the $\lambda_\mu(\alpha)$ are analytic for $|\alpha| < \varrho$. The principal term in (6) is identical with (2) if we put $z_0 = \frac{u(\alpha)}{\alpha}$, $c = 1 - \frac{1}{\sigma} \alpha^2 z_0^2 f''(\alpha z_0)$ since

$$-\int_0^\alpha \frac{u(t) - \sigma t}{t^2} dt = \frac{u(\alpha)}{\alpha} - f(u(\alpha)) - \sigma \log \frac{u(\alpha)}{\sigma\alpha} - \sigma.$$

To have a simple application put $\sigma = \frac{1}{2}$, $\tau = 1$, $f(z) = \frac{1}{4} z^2$, $u(t) = \frac{1 - \sqrt{1-t^2}}{t}$, $\alpha = (\cosh\varphi)^{-1}$. Obviously $A_k = \left(\frac{(2k+1)^{1/2}}{2\cosh\varphi}\right)^k H_k(x)$, $x = (2k+1)^{\frac{1}{2}} \cosh\varphi$, where $H_k(x)$ is the k-th Hermite polynomial. We have

$$-\int_0^\alpha \frac{u - \frac{1}{2}t}{t^2} dt = \frac{1}{2}\left(\frac{1}{\alpha^2}(1 - \sqrt{1-\alpha^2}) + \log\frac{1 + \sqrt{1-\alpha^2}}{2} - \frac{1}{2}\right) =$$

$$= \frac{1}{2}\left(\varphi + \frac{1}{2} e^{-2\varphi} - \log(2\cosh\varphi)\right),$$

$$u'(\alpha) = (1 - \sqrt{1-\alpha^2})\, \alpha^{-2} (1-\alpha^2)^{-\frac{1}{2}} = \frac{u(\alpha)}{\alpha} \coth\varphi;$$

hence from (5)

$$A_k \sim (2k+1)^k (2\sinh\varphi)^{-\frac{1}{2}} (2\cosh\varphi)^{-k} \exp\left\{\left(k + \frac{1}{2}\right)\left(\varphi + \frac{1}{2} e^{-2\varphi}\right)\right\} \times$$
$$\times (1 + k^{-1}\psi_1(\varphi) + k^{-2}\psi_2(\varphi) + \ldots),$$

$$e^{-\frac{1}{2}x^2} H_k(x) = 2^{\frac{k-1}{2}} \left(\frac{k}{e}\right)^{\frac{1}{2}k} (\sinh\varphi)^{-\frac{1}{2}} \exp\left\{\left(k + \frac{1}{2}\right)\left(\varphi - \frac{1}{2}\sinh 2\varphi\right)\right\} \times$$
$$\times \left(1 + \sum_{\mu=1}^{m} k^{-\mu} \psi_\mu(\varphi) + O(k^{-m-1})\right)$$

for $x = (2k+1)^{\frac{1}{2}} \cosh\varphi$, $\varphi \geq \varepsilon > 0$. This is a well known result due to PLANCHEREL and ROTACH[3]), who obtained it for real values of φ. The above proof is valid for any complex φ with $|\cosh\varphi| \geq \varepsilon$. The corresponding result for the "oscillating region" $x = (2n+1)^{\frac{1}{2}} \cos\varphi$, $\varepsilon \leq \varphi \leq \pi - \varepsilon$ cannot be obtained by our method.

There is another important application of Theorem 1 to which I intend to come back in another paper, namely, the problem of the asymptotic evaluation of certain partition functions such as the number of partitions of an integer n into exactly k parts. They depend on asymptotic developments of the same type as discussed in the present paper.

[3]) M. PLANCHEREL—W. ROTACH, Sur les valeurs asymptotiques des polynomes d'Hermite, *Commentarii Math. Helvetici*, **1** (1929), pp. 227—254. These authors use the method of steepest descent in their work. Regarding above form of the formula (with $m = 0$) see G. SZEGŐ, l. c., p. 195.

2. Proof of the theorem. We have from (1),

$$g'(z)=K\left(1-\sum_{\nu=1}^{\infty} c_\nu \alpha^\nu z^{\nu-1}\right)g(z)$$

hence

$$A_0=1,\ A_1=K(1-c_1\alpha),$$

(7) $$A_p=K(1-c_1\alpha)A_{p-1}-K\sum_{\nu=2}^{p}(p-1)\dots(p-\nu+1)c_\nu\alpha^\nu A_{p-\nu} \quad \text{for } p>1.$$

We assume that $A_p\neq 0$ for $p=1,\dots,k$ [4]) and write

(8) $$\varphi_s=s\frac{A_{s-1}}{A_s} \quad \text{for } s>0, \quad \varphi_s=0 \quad \text{for} \quad s\leqq 0.$$

Then

(9) $$A_p^{-1}=\frac{1}{p!}\varphi_1\dots\varphi_p$$

and by (7) and (8)

(10) $$\varphi_p=\frac{p}{K}+\sum_{\nu=1}^{\infty}c_\nu\alpha^\nu\varphi_p\dots\varphi_{p-\nu+1} \quad \text{for } p>0.$$

From (9), $-\log A_k=-\log k!+\sum_{p=1}^{k}\log\varphi_p$, hence

(11) $$-\frac{d}{d\alpha}(\log A_k(\alpha))=\sum_{p=1}^{k}\frac{1}{\varphi_p(\alpha)}\frac{d}{d\alpha}\varphi_p(\alpha).$$

The right hand side of (11) can be expressed in a very simple manner by $\varphi_k(\alpha)$ in consequence of the following differential — difference equation:

(12) $$\frac{t}{\varphi_p(t)}\frac{d}{dt}\varphi_p(t)=K(\varphi_p-\varphi_{p-1})-1, \quad p\geqq 1.$$

The formula is obviously true for $p=1$, since $\varphi_0(t)=0$, $\varphi_1(t)=\frac{1}{K}(1-c_1t)^{-1}$, hence we may assume its validity for $1\leqq i<p$. Summing (12) for $i=1,\dots,r$ we obtain

(13) $$\sum_{i=1}^{r}t\varphi_i^{-1}\varphi_i'=K\varphi_r-r \quad \text{for} \quad r<p.$$

Let us write $\pi_\nu=\prod_{i=1}^{\nu-1}\varphi_{p-i}$. Then from (10)

$$t\varphi_p^{-1}\varphi_p'=\sum_{\nu=1}^{p}\nu c_\nu t^\nu\pi_\nu+\sum_{\nu=1}^{p}c_\nu t^\nu\pi_\nu\sum_{i=0}^{\nu-1}t\varphi_{p-i}^{-1}\varphi_{p-i}'$$

$$=\sum_{\nu=1}^{p}\nu c_\nu t^\nu\pi_\nu+\sum_{\nu=1}^{p}c_\nu t^\nu\pi_\nu t\varphi_p^{-1}\varphi_p'+\sum_{\nu=1}^{p}c_\nu t^\nu\pi_\nu\left(K(\varphi_{p-1}-\varphi_{p-\nu})-(\nu-1)\right)$$

by the induction hypothesis, hence

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[4]) This assumption in convenient but not really essential for the proof. We can always enforce the condition by slightly changing the value of the parameter α.

$$\left(1-\sum_{\nu=1}^{p} c_\nu t^\nu \pi_\nu\right) t\varphi_p^{-1}\varphi_p' = \sum_{\nu=1}^{p} c_\nu t^\nu \pi_\nu (K(\varphi_{p-1}-\varphi_{p-\nu})+1) =$$

$$= \sum_{\nu=1}^{p} c_\nu t^\nu \pi_\nu (K(\varphi_p-\varphi_{p-\nu}) - K(\varphi_p - \varphi_{p-1})+1) =$$

$$= K\sum_{\nu=1}^{p} c_\nu t^\nu (\varphi_p\varphi_{p-1}\ldots\varphi_{p-\nu+1} - \varphi_{p-1}\ldots\varphi_{p-\nu}) - \sum_{\nu=1}^{p} c_\nu t^\nu \pi_\nu (K(\varphi_p-\varphi_{p-1})-1) =$$

$$= K\left[\left(\varphi_p - \frac{p}{K}\right) - \left(\varphi_{p-1} - \frac{p-1}{K}\right)\right] - \sum_{\nu=1}^{p} c_\nu t^\nu \pi_\nu (K(\varphi_p - \varphi_{p-1})-1) =$$

$$= \left(1-\sum_{\nu=1}^{p} c_\nu t^\nu \pi_\nu\right)(K(\varphi_p-\varphi_{p-1})-1)$$

whence our assertion follows $\left(\text{since } (1-\sum_{\nu=1}^{p} c_\nu t^\nu \pi_\nu)\,\varphi_p = \frac{p}{K} \neq 0\right)$. Hence (13) is valid for $r=p$, $\sum_{i=1}^{p} \varphi_i^{-1}\varphi_i' = \frac{K}{t}\left(\varphi_p - \frac{p}{K}\right)$ for $p=1, 2,\ldots$ and $-\frac{d}{d\alpha}\log A_k(\alpha)$ $= \frac{K}{\alpha}\left(\varphi_k(\alpha) - \frac{k}{K}\right)$ by (11). Integrating and noting that, by (9), $\varphi_p(0) = \frac{p}{K}$ and $A_k(0) = K^k$, we obtain

$$A_k = K^k \exp\left\{-K\int_0^\alpha \left(\varphi_k(t) - \frac{k}{K}\right) t^{-1}\,dt\right\}. \tag{14}$$

This relation reduces the problem to the asymptotic evaluation of $\varphi_k(t)$. Write $\varphi_p(t) = \sum_{\nu=0}^{\infty} d_p(\nu) t^\nu$ where $d_p(\nu)=0$ for $p \leq 0$, $d_p(0) = \frac{p}{K}$ for $p>0$, and

$$d_p(1)t + d_p(2)t^2 + \ldots = \sum_{\nu=1}^{\infty} c_\nu t^\nu \prod_{r=1}^{\nu}\left(\sum_{i=0}^{\infty} d_{p-r+1}(i) t^i\right) \text{ for } p>0. \tag{15}$$

The idea naturally suggests itself to compare $\varphi_p(t)$ with the function $\varphi = \varphi\left(\frac{p}{K}, t\right)$ defined by the equation $\varphi = \frac{p}{K} + \sum_{\nu=1}^{\infty} c_\nu t^\nu \varphi^\nu$. Generally let

$$\varphi(\xi, t) = \sum_{\nu=0}^{\infty} d(\xi, \nu)\, t^\nu, \quad d(\xi, 0) = \xi \tag{16}$$

be defined for $\xi > 0$ by

$$\varphi(\xi, t) = \xi + \sum_{\nu=1}^{\infty} c_\nu t^\nu \varphi^\nu(\xi, t) = \xi + t\varphi(\xi, t) f'(t\varphi(\xi, t)). \tag{17}$$

From (16) we have

$$d(\xi, 1)t + d(\xi, 2)t^2 + \ldots = \sum_{\nu=1}^{\infty} c_\nu t^\nu [\xi + d(\xi, 1)t + d(\xi, 2)t^2 + \ldots]^\nu \tag{18}$$

whence $d(\xi, 1) = c_1\xi$, $d(\xi, 2) = c_2\xi^2 + c_1^2\xi, \ldots,$

$$d(\xi, \nu) = \sum_{r_1+\ldots+r_i=\nu} b(r_1, \ldots, r_i)\, c_{r_1}\ldots c_{r_i}\, \xi^{\nu-i+1} \quad \text{for } \nu > 0 \tag{19}$$

where the $b(r_1, \ldots, r_i)$ are positive absolute constants and the summation runs over the (unrestricted) partitions of ν.

Comparing (17) and (3) we see that

(20) $u(t) = t\varphi(\sigma, t), \quad d_\nu = d(\sigma, \nu) = \sum_{r_1+\cdots+r_i=\nu} b(r_1, \ldots, r_i)\, c_{r_1} \ldots c_{r_i}\, \sigma^{\nu-i+1}$

and $|d_\nu| \leqq \sum b(r_1, \ldots, r_i)\, |c_{r_1} \ldots c_{r_i}|\, \sigma^{\nu-i+1} = D_\nu$. This shows that $|d(\xi, \nu)| \leqq D_\nu$ if $\xi \leqq \sigma$ and $|d(\xi, \nu)| \leqq \left(\frac{\xi}{\sigma}\right)^\nu D_\nu$ if $\xi > \sigma$. Hence, if $\varrho(\xi)$ denotes the radius of convergence of (16), then $\varrho(\xi) \geqq \varrho$ if $\xi \leqq \sigma$ and $\varrho(\xi) \geqq \varrho \frac{\sigma}{\xi}$ if $\xi > \sigma$. In particular

$$\varrho\left(\frac{k}{K}\right) = \varrho\left(\frac{k}{\frac{1}{\sigma}k + \tau}\right) = \varrho\left(\left(1 - \frac{\tau}{\frac{1}{\sigma}k + \tau}\right)\sigma\right) \geqq \left(1 - \frac{\sigma|\tau|}{k}\right)\varrho,$$

and $\varphi\left(\frac{k}{K}, t\right)$ is certainly convergent for sufficiently large k if $|t| \leqq \varrho_0 < \varrho$. (We only have to take $k > \frac{\varrho}{\varrho - \varrho_0}\sigma|\tau|$.) It follows that $\varphi\left(\frac{p}{K}, \alpha\right)$ is convergent for every $p \leqq k$. Comparing the coefficients $d_p(\nu)$ and $d\left(\frac{p}{K}, \nu\right)$ for $p \leqq k$, $\nu = 0, 1, \ldots$ we find from (15) and (18) $d_p(0) = d\left(\frac{p}{K}, 0\right) = \frac{p}{K}$,

$$d_p(1) = d\left(\frac{p}{K}, 1\right) = c_1\frac{p}{K}, \quad d_p(2) = \frac{p}{K}c_1^2 + \frac{p(p-1)}{K^2}c_2 = d\left(\frac{p}{K}, 2\right) - \frac{p}{K^2}c_2,$$

$d_p(\nu) = d\left(\frac{p}{K}, \nu\right) + O\left(\frac{h(\nu)}{K}\right)$ for every fixed ν. This shows that the error commited by taking $\varphi\left(\frac{k}{K}, t\right)$ instead of $\varphi_k(t)$ is small; in fact, $\varphi_r(\alpha) = \varphi\left(\frac{p}{K}, \alpha\right) + O\left(\frac{1}{K}\right)$ provided that $h(\nu)$ is not increasing too rapidly with ν. In order to have an estimate for $h(\nu)$ we introduce the following notation:

For $r > 0$, $i \geqq 0$ write

$$(1 - ix)(1 - (i+1)x) \ldots (1 - (i+r-1)x) \equiv \sum_{\nu=0}^{r} (-1)^\nu S(i, r; \nu) x^\nu,$$

where $S(i, r; 0) = 1$, and denote by $P_p(i, r)$, $p > i$ an arbitrary expression

(21) $$P_p(i, r) = \sum_{\nu=0}^{\infty} (-1)^\nu \alpha_\nu p^{-\nu}$$

where $\alpha_0 = 1$ and the coefficients α_ν satisfy for $\nu > 0$

(22) $$0 \leqq \alpha_\nu \leqq S(i, r; \nu)$$

(hence $\alpha_\nu = 0$ for $\nu > r$), and

(23) $$\sum_{\nu=0}^{\infty} (-1)^\nu \alpha_{s+\nu} p^{-s-\nu} \geqq 0 \quad \text{for } s = 0, 1, \ldots, r.$$

The second condition clearly implies

(24) $$\sum_{\nu=0}^{\infty}(-1)^{\nu}\alpha_{s+\nu}p^{-s-\nu}\leqq\alpha_s p^{-s} \quad \text{for} \quad s=0,\ldots,r.$$

If $a_1>0$, $a_2>0$ then

(25) $$a_1P_p(i,r)+a_2P_p(i,r)=(a_1+a_2)\,P_p(i,r).$$

In this formula (like in others which follow) each $P_p(i,r)$ may denote a different expression of the form (21). The formula is to be read from the left to right: if $a_1>0$, $a_2>0$ and both expressions $P_p(i,r)$ on the left satisfy (22) and (23) then the $P_p(i,r)$ on the right (which is uniquely determined by the left hand side) also satisfies these conditions. The proof is obvious

The following formula is less trivial:

(26) $$P_p(i,r)\,P_p(j,s)=P_p(i,r+s) \quad \text{if} \quad j\leqq i+r.$$

Write $P_p(i,r)=\sum(-1)^{\nu}\alpha_\nu p^{-\nu}$, $P_p(j,s)=\sum(-1)^{\nu}\beta_\nu p^{-\nu}$ and $P_p(i,r)\,P_p(j,s)=\sum(-1)^{\nu}\gamma_\nu p^{-\nu}$ where $\gamma_\nu=\alpha_0\beta_\nu+\ldots+\alpha_\nu\beta_0$.

Comparing $(1-ix)\ldots(1-(i+r-1)x)(1-jx)\ldots(1-(j+s-1)x)$ with $(1-ix)\ldots(1-(i+r+s-1)x)$ we see that $\sum_{\mu=0}^{\nu}S(i,r;\mu)\,S(j,s;\nu-\mu)\leqq$ $\leqq S(i,r+s;\nu)$ if $j\leqq i+r$ hence $\gamma_\nu=\alpha_0\beta_\nu+\ldots+\alpha_\nu\beta_0\leqq S(i,r+s;\nu)$ which proves condition (22). Condition (23) is a consequence of the following

Lemma. *Let $\{a_0,a_1,\ldots\}$, $\{b_0,b_1,\ldots\}$ be non-negative sequences with finite sums. Let us form the alternating series $\sum(-1)^{\nu}a_\nu$, $\sum(-1)^{\nu}b_\nu$ and suppose that the remainders of the two series themselves form an alternating sequence, i. e. $a_\nu-a_{\nu+1}+a_{\nu+2}-\ldots\geqq0$, $b_\nu-b_{\nu+1}+b_{\nu+2}-\ldots\geqq0$ for $\nu\geqq0$. Then the remainders of the Cauchy product $c_\nu=a_0b_\nu+\ldots+a_\nu b_0$ also form an alternating sequence: $c_\nu-c_{\nu+1}+c_{\nu+2}-\ldots\geqq0$.*

The lemma is notably true if both $\{a_\nu\}$ and $\{b_\nu\}$ are finite.

Proof. $$\sum_{r=0}^{\infty}(-1)^{r}c_{\nu+r}=\left(\sum_{r=0}^{\infty}(-1)^{r}a_r\right)\left(\sum_{r=0}^{\infty}(-1)^{r}b_{\nu+r}\right)+$$
$$+b_{\nu-1}\sum_{r=0}^{\infty}(-1)^{r}a_{1+r}+b_{\nu-2}\sum_{r=0}^{\infty}(-1)^{r}a_{2+r}+\ldots+b_0\sum_{r=0}^{\infty}(-1)^{r}a_{\nu+r}\geqq0$$

since each term is $\geqq0$.

Applying the lemma to $a_\nu=\alpha_\nu p^{-\nu}$, $b_\nu=\beta_\nu p^{-\nu}$, $c_\nu=\gamma_\nu p^{-\nu}$, we immediately obtain condition (23) for the right hand side of (26). A repeated application of the lemma also shows that

$$\left(1-\frac{i}{p}\right)\ldots\left(1-\frac{i+r-1}{p}\right)\equiv\sum_{\nu=0}^{r}(-1)^{\nu}p^{-\nu}S(i,r;\nu)=P_p(i,r).$$

We shall now show that for $p > i \geqq 0$, $\nu > 0$

$$(27)\qquad d_{p-i}(\nu) = \sum b(r_1, \ldots, r_i)\, c_{r_1} \ldots c_{r_i} \left(\frac{p}{K}\right)^{\nu-i+1} P_p(i, \nu)$$

summed for the partitions $r_1 + \ldots + r_i = \nu$. For the expression on the right we use the shorthand notation $d\left(\frac{p}{K}, \nu\right) \times P_p(i, \nu)$, the symbol $\times$ reminding us of the fact that $d\left(\frac{p}{K}, \nu\right)$ is a sum of terms (19) each term being multiplied by a $P_p(i, \nu)$.

Writing $p-i$ instead of p in (15), we obtain by comparing the coefficients of t, $d_{p-i}(1) = c_1 \frac{p-i}{K} = c_1 \frac{p}{K}\left(1 - \frac{i}{p}\right) = d\left(\frac{p}{K}, 1\right) P_p(i, 1)$ for $i < p$ which proves (27) for $\nu = 1$. We note that $d_{p-i}(0) = \frac{p-i}{K} = d\left(\frac{p}{K}, 0\right) P_p(i, 1)$, hence assuming the induction hypothesis,

$$(28)\qquad d_{p-i}(r) = d\left(\frac{p}{K}, r\right) \times P_p(i, r+1) \quad \text{for} \quad i < p, \quad 0 \leqq r < \nu.\ ^{5)}$$

Equating the coefficients of t^ν in (15) and writing again $p-i$ in the place of p, we obtain

$$(29)\qquad d_{p-i}(\nu) = \sum_{\mu=1}^{\nu} c_\mu \sum_{s_1 + \ldots + s_\mu = \nu-\mu} d_{p-i}(s_1)\, d_{p-i-1}(s_2) \ldots d_{p-i-\mu+1}(s_\mu),$$

the summation extending over every composition of $\nu - \mu$ with $s_1, \ldots, s_\mu \geqq 0$. Since $s_1 + \ldots + s_\mu < \nu$, we have by (28) for every non-zero term of the sum

$$(30)\qquad d_{p-i}(s_1) \ldots d_{p-i-\mu+1}(s_\mu) =$$
$$= d\left(\frac{p}{K}, s_1\right) \ldots d\left(\frac{p}{K}, s_\mu\right) \times P_p(i, s_1+1)\, P_p(i+1, s_2+1) \ldots P_p(i+\mu-1, s_\mu+1) =$$
$$= d\left(\frac{p}{K}, s_1\right) \ldots d\left(\frac{p}{K}, s_\mu\right) \times P_p(i, \nu)$$

since

$$P_p(i, s_1+1)\, P_p(i+1, s_2+1) \ldots P_p(i+\mu-1, s_\mu+1) =$$
$$= P_p(i, s_1+s_2+2)\, P_p(i+2, s_3+1) \ldots P_p(i+\mu-1, s_\mu+1) =$$
$$= \ldots = P_p(i, s_1+s_2+\ldots+s_\mu+\mu) = P_p(i, \nu)$$

by a repeated application of (26). The same is true if $p - i - \mu + 1 \leqq 0$, that is, the left hand side of (30) is zero, since then $p < i + \mu \leqq i + \nu$ and obviously $0 = P_p(i, \nu)$ if $i < p < i + \nu$. Now from (18) we have for $\xi = p/K$

$$(31)\qquad d\left(\frac{p}{K}, \nu\right) = \sum_{\mu=1}^{\nu} c_\mu \sum d\left(\frac{p}{K}, s_1\right) \ldots d\left(\frac{p}{K}, s_\mu\right)$$

where the summation is to be taken for the same compositions as in (29).

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5) Here we have used the obvious equation $P_p(i, r) = P_p(i, r+1)$, which should be read from the left to the right.

This shows that to each term $T=c_\mu d\left(\frac{p}{K}, s_1\right)\dots d\left(\frac{p}{K}, s_\mu\right)$ in (31) there is a corresponding term T^* in (29) with $T^* = T\times P_p(i,\nu)$ by (30). The precise meaning of this relation is that if we substitute (19) $\left(\text{with } \xi=\frac{p}{K}\right)$ for each $d\left(\frac{p}{K}, s_i\right)$ in T and carry out term-by-term multiplication then each of these terms appears in T^* multiplied by a $P_p(i,\nu)$. Collecting terms belonging to the same $c_{r_1}\dots c_{r_i}$ we have by (25), since the multiplying constants $b(r_1,\dots,r_i)$ are positive, $d_{p-i}(\nu)=d\left(\frac{p}{K},\nu\right)\times P_p(i,\nu)$, i. e. (27). In particular, $d_p(\nu)=d\left(\frac{p}{K},\nu\right)\times P_p(0,\nu)$ and $d_k(\nu)=d\left(\frac{k}{K},\nu\right)\times P_k(0,\nu)$. Therefore

$$\varphi_k(t)=\sum_{\nu=0}^{\infty} d_k(\nu)t^\nu=\sum_{\nu=0}^{\infty} d\left(\frac{k}{K},\nu\right)\times P_k(0,\nu)\,t^\nu=$$

$$=\sum_{\nu=0}^{\infty}\sum_{r_1+\dots+r_i=\nu} b(r_1,\dots,r_i)\,c_{r_1}\dots c_{r_i}\left(\frac{k}{K}\right)^{\nu-i+1}(1-k^{-1}\beta(r_1,\dots,r_i;1)+ \tag{32}$$

$$+\dots+(-1)^\nu k^{-\nu}\beta(r_1,\dots,r_i;\nu))\,t^\nu$$

where

$$0\leqq\beta(r_1,\dots,r_i;j)\leqq S(0,\nu;j)<\nu^{2j}, \tag{33}$$

$$0\leqq 1-k^{-1}\beta(r_1,\dots,r_i;1)+\dots+(-1)^\nu k^{-\nu}\beta(r_1,\dots,r_i;\nu)\leqq 1. \tag{34}$$

This shows first, that the radius of $\varphi_k(t)$ is not less than $\left(1-\frac{\sigma|\tau|}{k}\right)\varrho$, hence $\varphi_k(\alpha)$ is convergent if k is sufficiently large. Secondly,

$$\varphi_k(t)=\varphi\left(\frac{k}{K},t\right)+ \tag{35}$$

$$+\sum_{\mu=1}^{m}(-1)^\mu k^{-\mu}\sum_{\nu=\mu}^{\infty}\left(\sum_{r_1+\dots+r_i=\nu} b(r_1,\dots,r_i)\,c_{r_1}\dots c_{r_i}\left(\frac{k}{K}\right)^{\nu-i+1}\beta(r_1,\dots,r_i;\mu)\right)t^\nu+$$

$$+\sum_{\nu=m+1}^{\infty}\left\{\sum_{r_1+\dots+r_i=\nu} b(r_1,\dots,r_i)\,c_{r_1}\dots c_{r_i}\left(\frac{k}{K}\right)^{\nu-i+1}\left(\sum_{\mu=m+1}^{\nu}(-1)^\mu k^{-\mu}\beta(r_1,\dots,r_i;\mu)\right)\right\}t^\nu$$

for $|t|\leqq\varrho_0$. For, each of the series

$$\sum_{\nu=\mu}^{\infty}\left(\sum b(r_1,\dots,r_i)\,c_{r_1}\dots c_{r_i}\left(\frac{k}{K}\right)^{\nu-i+1}\beta(r_1,\dots,r_i;\mu)\right)t^\nu,\quad \mu=1,\dots,m,$$

is absolutely convergent for $|t|\leqq\varrho_0$ by (33), and also the remainder since

$$\left|\sum_{\mu=m+1}^{\nu}(-1)^\mu k^{-\mu}\beta(r_1,\dots,r_i;\mu)\right|\leqq k^{-m-1}\beta(r_1,\dots,r_i;m+1)<k^{-m-1}\nu^{2m+2}$$

by (24). Hence the remainder term in (35) can be written in the form $k^{-m-1}t\chi(t)$, where $\chi(t)$ is absolutely convergent for $|t|\leqq\varrho_0$.

Finally we replace $\frac{k}{K}$ by σ, $\varphi\left(\frac{k}{K},t\right)$ by $\varphi(\sigma,t)$ in (35) by putting

$$\frac{k}{K}=\frac{k}{\frac{1}{\sigma}k+\tau}=\sigma\left(1-\frac{\sigma\tau}{k}+\ldots+(-1)^m\frac{\sigma^m\tau^m}{k^m}+(-1)^{m+1}\frac{\sigma^{m+1}\tau^{m+1}}{k^m(k+\sigma\tau)}\right)\quad\text{and}$$

collecting terms belonging to equal powers of k. As a result we obtain the asymptotic development

$$(36)\qquad t^{-1}\left(\varphi_k(t)-\frac{k}{K}\right)=t^{-1}(\varphi(\sigma,t)-\sigma)+\sum_{\mu=1}^{m}k^{-\mu}\psi_\mu^*(t)+O(k^{-m-1})$$

valid for $|t|\leq\varrho_0$, hence for $t=\alpha$.

(14), (36) and (20) give

$$(37)\quad A_k=K^k\exp\left\{-K\int_0^\alpha\frac{u-\sigma t}{t^2}\,dt-\frac{1}{\sigma}\int_0^\alpha\psi_1^*(t)\,dt-\right.$$

$$\left.-\sum_{\mu=1}^{m-1}k^{-\mu}\int_0^\alpha\left(\tau\psi_\mu^*+\frac{1}{\sigma}\psi_{\mu+1}^*\right)dt+O(k^{-m})\right\}.$$

It remains to determine the explicit expression for $\psi_1^*(t)$. We have from (36), if we put $u(t)=t\varphi(\sigma,t)$ and $\psi(t)=-t\psi_1^*(t)+\sigma^2\tau$,

$$\varphi_k(t)\sim\frac{k}{K}-\sigma+\frac{u(t)}{t}+\frac{t}{k}\psi_1^*(t)\sim\frac{u(t)}{t}-\frac{1}{k}\psi(t)$$

where $\sim$ indicates that the difference of the left and right hand sides is $O(k^{-2})$. Using equation (12) we obtain

$$\varphi_{k-1}=\varphi_k\quad-\frac{1}{K}(1+t\varphi_k^{-1}\varphi_k')\sim\frac{u}{t}-\frac{1}{k}\psi-\frac{\sigma}{k}t\frac{u'}{u},$$

$$\varphi_{k-2}=\varphi_{k-1}-\frac{1}{K}(1+t\varphi_{k-1}^{-1}\varphi_{k-1}')\sim\frac{u}{t}-\frac{1}{k}\psi-\frac{2\sigma}{k}t\frac{u'}{u},$$

generally $\varphi_{k-i}\sim\frac{u}{t}-\frac{1}{k}\psi-\frac{i\sigma}{k}t\frac{u'}{u}$. We can put these expressions, purely formally, into (10) since we know that the asymptotic expansion (36) is valid

$$\frac{u}{t}-\frac{1}{k}\psi\sim\frac{k}{K}+\sum c_\nu t^\nu\prod_{i=0}^{\nu-1}\left(\frac{u}{t}-\frac{1}{k}\psi-\frac{i\sigma}{k}t\frac{u'}{u}\right)$$

$$\sim\sigma-\frac{\sigma^2\tau}{k}+\sum_{\nu=1}^{\infty}c_\nu u^\nu-\frac{1}{k}\sum_{\nu=1}^{\infty}\nu c_\nu t u^{\nu-1}\psi-\frac{\sigma}{k}\sum_{\nu=1}^{\infty}\binom{\nu}{2}c_\nu t^2u^{\nu-2}u'$$

whence

$$\left(1-\sum_{\nu=1}^{\infty}\nu c_\nu t u^{\nu-1}\right)\psi=\sigma^2\tau+\sigma\sum_{\nu=1}^{\infty}\binom{\nu}{2}c_\nu t^2u^{\nu-2}u'.$$

From (3), $u'=\frac{u}{t}+t\sum\nu c_\nu u^{\nu-1}u'$, $\quad 1-\sum\nu c_\nu t u^{\nu-1}=\frac{u}{u't}$,

$$\sum\binom{\nu}{2}c_\nu t^2u^{\nu-2}u'=\frac{u}{u't}-1+\frac{1}{2}\frac{uu''}{(u')^2},\ \text{hence}\ \frac{1}{\sigma}\psi=(\sigma\tau-1)t\frac{u'}{u}+1+\frac{1}{2}t\frac{u''}{u'},$$

$$-\frac{1}{\sigma}\psi_1^*=\frac{1}{t}\left(\frac{1}{\sigma}\psi-\sigma\tau\right)=(\sigma\tau-1)\left(\frac{u'}{u}-\frac{1}{t}\right)+\frac{1}{2}\frac{u''}{u'},$$

$$-\frac{1}{\sigma}\int_0^\alpha\psi_1^*(t)\,dt=(\sigma\tau-1)\log\frac{u(\alpha)}{\sigma\alpha}+\frac{1}{2}\log\frac{u'(\alpha)}{\sigma}.$$

Theorem 1 follows from this and (37).

It would be possible to obtain, in a similar manner, the next term $\psi_2^*(t)$, but the general term $\psi_\mu^*(t)$ could hardly be obtained in that way explicitely. The method of steepest descent gives heuristically, in the case $\sigma=1$, $\tau=1$, the following expressions for $\psi_\mu(\alpha)$ in

$$A_{K-1}\sim K^{K-1}(u'(\alpha))^{\frac{1}{2}}\exp\left\{-K\int_0^\alpha \frac{u-t}{t^2}dt\right\}\left(1+\sum_{\mu=1}^{\infty}K^{-\mu}\psi_\mu(\alpha)\right).$$

Put
$$b_\nu(\alpha)=\frac{(-1)^{\nu-1}}{\nu}\left(\frac{\alpha}{u(\alpha)}\right)^\nu+\frac{\alpha^\nu}{\nu!}f^{(\nu)}(u(\alpha)),$$

$$\exp\{\xi\sum_{\nu=1}^{\infty}b_{\nu+2}(\alpha)\zeta^\nu\}\equiv\sum_{\mu=0}^{\infty}h_\mu(\xi,\alpha)\zeta^\mu;\ h_\mu(\xi,\alpha)=\sum_{\nu=0}^{\mu}a_{\mu\nu}(\alpha)\,\xi^\nu,$$

then

$$\psi_\mu(\alpha)\equiv\sum_{\nu=0}^{2\mu}\frac{\Gamma\left(\mu+\nu+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}a_{2\mu,\nu}(\alpha)\,(2u'(\alpha))^{\nu+\mu}.$$

3. The following additional remarks might widen the field of applicability of Theorem 1. Throughout the previous proof, α was considered a fixed parameter. Now it is clear that nothing will change in the proof if we assume that α is not fixed but depends on k, of course subject to the condition $|\alpha|\leqq\varrho_0<\varrho$. In particular, Theorem 1 remains true if α tends to 0 as $k\to\infty$. For example, the formula of PLANCHEREL and ROTACH for Hermite polynomials holds uniformly if $\varphi\to\infty$ as $k\to\infty$.

If we are interested in the asymptotic behaviour of A_k when k tends less rapidly to ∞ than K, we have to assume that σ is not a constant but tends to 0 as $k\to\infty$. In that case it is preferable to put $\tau=0$, $\sigma=\sigma(k)=\frac{k}{K}$ so that $u(t)=u(\sigma,t)$ depends on k. Again, Theorem 1 remains true, with the additional remark that the $\psi_\mu(\alpha)$ now depend on k and tend to 0 as $k\to\infty$. As a matter of fact, we can always put $\psi_\mu(\alpha)=\frac{k}{K}\chi_\mu(\alpha)$ as readily seen from (35).

Finally let us consider a particular case when the function $f(z)$ itself depends on k. We assume that the coefficients c_ν have an asymptotic development of the form[6])

$$c_\nu=C_\nu\left(1+\sum_{\mu=1}^{m-1}\binom{\nu+1}{\mu}E_\mu K^{-\mu}+\delta_\nu E_m\binom{\nu+1}{m}K^{-m}\left(1+\frac{a}{K}\right)^\nu\right) \tag{38}$$

where C_μ, E_μ, $a\geqq 0$ are constants (not depending on k), and $|\delta_\nu|<1$. Let m be fixed and put $M=\max(|E_\mu|^{\frac{1}{\mu}})$, $\mu=1,\ldots,m$, so that

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[6]) This case is important from the point of view of partitions to which I have referred above.

$\left|\binom{\nu+1}{\mu} E_\mu\right| \leqq \binom{\nu+1}{\mu} M^\mu =$ the coefficient of $K^{-\mu}$ in $\left(1+\frac{M}{K}\right)^{\nu+1}$. Then if $r_1+\ldots+r_i=\nu$,

$$(39)\quad c_{r_1}\ldots c_{r_i} = C_{r_1}\ldots C_{r_i}\prod_{j=1}^{i}\left(1+\sum_{\mu=1}^{m-1}\binom{r_j+1}{\mu}E_\mu K^{-\mu}+\delta E_m\binom{r_j+1}{m}K^{-m}\left(1+\frac{a}{K}\right)^{r_j}\right)$$
$$= C_{r_1}\ldots C_{r_i}\left(1+\sum_{\mu=1}^{m-1}E(r_1,\ldots,r_i;\mu)K^{-\mu}+\delta E(r_1,\ldots,r_i;m)K^{-m}\left(1+\frac{a}{K}\right)^{\nu}\right)$$

where $|\delta|<1$ and

$$(40)\quad |E(r_1,\ldots,r_i;\mu)| < \binom{\nu+i}{\mu}M^\mu \leqq \binom{2\nu}{\mu}M^\mu = O(\nu^\mu).$$

Let us write $U(t)$, $V(t)$ for the u, v-functions belonging to $F(z) = \sum_{\nu=1}^{\infty}\frac{1}{\nu}C_\nu z^\nu$ and $\bar{\varrho}$ for the radius of $V(t)$, then $\varrho \geqq \bar{\varrho}\left(1+\frac{a}{K}\right)^{-1}$ by (19), (39) and (40), hence $\varphi(\sigma,t)$ is convergent for $|t| \leqq \varrho_0 < \varrho$ if k is large. Also the expansion (36) remains valid as seen by putting (39) into (35) and noticing that the power series belonging to a fixed $k^{-\mu}$, $\mu=1,\ldots,m$, is absolutely convergent by (40).

Theorem 2. *If in Theorem* 1, *the coefficients* c_ν *have an asymptotic development* (38) *and we put* $F(z)=\sum_{\nu=1}^{\infty}\frac{1}{\nu}C_\nu z^\nu$, $U=t\left(\sigma+\sum_{\nu=1}^{\infty}C_\nu U^\nu\right)$, $V=t\left(\sigma+\sum_{\nu=1}^{\infty}|C_\nu|V^\nu\right)$ *then*

$$\log A_k = k\log K - K\int_0^\alpha \frac{U(t)-\sigma t}{t^2}\,dt + \sum_{\mu=0}^{m}k^{-\mu}\psi_\mu(\alpha) + O(k^{-m-1})$$

uniformly for $|\alpha| \leqq \varrho_0 < \bar{\varrho}$ *if* $\bar{\varrho}$ *is the radius of* $V(t)$. *Also*

$$\psi_0(\alpha) = (\sigma\tau-1)\log\frac{U(\alpha)}{\sigma\alpha} + \frac{1}{2}\log\left(\frac{1}{\sigma}U'(\alpha)\right)$$

if $E_1=0$ *in* (38).

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